

## thy.1 Computably Enumerable Sets not Closed under Complement

cmp:thy:cmp:sec Suppose  $A$  is computably enumerable. Is the complement of  $A$ ,  $\bar{A} = \mathbb{N} \setminus A$ , necessarily computably enumerable as well? The following theorem and corollary show that the answer is “no.”

cmp:thy:cmp:thm:ce-comp **Theorem thy.1.** *Let  $A$  be any set of natural numbers. Then  $A$  is computable if and only if both  $A$  and  $\bar{A}$  are computably enumerable.*

*Proof.* The forwards direction is easy: if  $A$  is computable, then  $\bar{A}$  is computable as well ( $\chi_A = 1 - \chi_{\bar{A}}$ ), and so both are computably enumerable.

In the other direction, suppose  $A$  and  $\bar{A}$  are both computably enumerable. Let  $A$  be the domain of  $\varphi_d$ , and let  $\bar{A}$  be the domain of  $\varphi_e$ . Define  $h$  by

$$h(x) = \mu s (T(d, x, s) \vee T(e, x, s)).$$

In other words, on input  $x$ ,  $h$  searches for either a halting computation of  $\varphi_d$  or a halting computation of  $\varphi_e$ . Now, if  $x \in A$ , it will succeed in the first case, and if  $x \in \bar{A}$ , it will succeed in the second case. So,  $h$  is a total computable function. But now we have that for every  $x$ ,  $x \in A$  if and only if  $T(e, x, h(x))$ , i.e., if  $\varphi_e$  is the one that is defined. Since  $T(e, x, h(x))$  is a computable relation,  $A$  is computable.  $\square$

It is easier to understand what is going on in informal computational terms: explanation to decide  $A$ , on input  $x$  search for halting computations of  $\varphi_e$  and  $\varphi_f$ . One of them is bound to halt; if it is  $\varphi_e$ , then  $x$  is in  $A$ , and otherwise,  $x$  is in  $\bar{A}$ .

cmp:thy:cmp:cor:comp-k **Corollary thy.2.**  *$\bar{K}_0$  is not computably enumerable.*

*Proof.* We know that  $K_0$  is computably enumerable, but not computable. If  $\bar{K}_0$  were computably enumerable, then  $K_0$  would be computable by **Theorem thy.1**.  $\square$

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## Bibliography