

thy.1 Computably Enumerable Sets are Closed under Union and Intersection

cmp:thy:clo:
sec The following theorem gives some closure properties on the set of computably enumerable sets.

Theorem thy.1. *Suppose A and B are computably enumerable. Then so are $A \cap B$ and $A \cup B$.*

Proof. μy allows us to use various characterizations of the computably enumerable sets. By way of illustration, we will provide a few different proofs.

For the first proof, suppose A is enumerated by a computable function f , and B is enumerated by a computable function g . Let

$$\begin{aligned} h(x) &= \mu y (f(y) = x \vee g(y) = x) \text{ and} \\ j(x) &= \mu y (f((y)_0) = x \wedge g((y)_1) = x). \end{aligned}$$

Then $A \cup B$ is the domain of h , and $A \cap B$ is the domain of j .

Here is what is going on, in computational terms: given procedures that enumerate A and B , we can semi-decide if an element x is in $A \cup B$ by looking for x in either enumeration; and we can semi-decide if an element x is in $A \cap B$ by looking for x in both enumerations at the same time. explanation

For the second proof, suppose again that A is enumerated by f and B is enumerated by g . Let

$$k(x) = \begin{cases} f(x/2) & \text{if } x \text{ is even} \\ g((x-1)/2) & \text{if } x \text{ is odd.} \end{cases}$$

Then k enumerates $A \cup B$; the idea is that k just alternates between the enumerations offered by f and g . Enumerating $A \cap B$ is trickier. If $A \cap B$ is empty, it is trivially computably enumerable. Otherwise, let c be any element of $A \cap B$, and define l by

$$l(x) = \begin{cases} f((x)_0) & \text{if } f((x)_0) = g((x)_1) \\ c & \text{otherwise.} \end{cases}$$

In computational terms, l runs through pairs of elements in the enumerations of f and g , and outputs every match it finds; otherwise, it just stalls by outputting c .

For the last proof, suppose A is the *domain* of the partial function $m(x)$ and B is the domain of the partial function $n(x)$. Then $A \cap B$ is the domain of the partial function $m(x) + n(x)$.

In computational terms, if A is the set of values for which m halts and B is the set of values for which n halts, $A \cap B$ is the set of values for which both procedures halt. explanation

Expressing $A \cup B$ as a set of halting values is more difficult, because one has to simulate m and n in parallel. Let d be an index for m and let e be an

index for n ; in other words, $m = \varphi_d$ and $n = \varphi_e$. Then $A \cup B$ is the domain of the function

$$p(x) = \mu y (T(d, x, y) \vee T(e, x, y)).$$

explanation In computational terms, on input x , p searches for either a halting computation for m or a halting computation for n , and halts if it finds either one. \square

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Bibliography