und.1 The Decision Problem is Unsolvable

tms:und:uns:sec

tms:und:uns:thm:decision-prob **Theorem und.1.** *The decision problem is unsolvable.*

Proof. Suppose the decision problem were solvable, i.e., suppose there were a Turing machine D of the following sort. Whenever D is started on a tape that contains a sentence ψ of first-order logic as input, D eventually halts, and outputs 1 iff ψ is valid and 0 otherwise. Then we could solve the halting problem as follows. We construct a Turing machine E that, given as input the number e of Turing machine M_e and input w, computes the corresponding sentence $\tau(M_e, w) \rightarrow \alpha(M_e, w)$ and halts, scanning the leftmost square on the tape. The machine $E \frown D$ would then, given input e and w, first compute $\tau(M_e, w) \rightarrow \alpha(M_e, w)$ and then run the decision problem machine D on that input. D halts with output 1 iff $\tau(M_e, w) \rightarrow \alpha(M_e, w)$ is valid and outputs 0 otherwise. By **??** and **??**, $\tau(M_e, w) \rightarrow \alpha(M_e, w)$ is valid iff M_e halts on input w. Thus, $E \frown D$, given input e and w halts with output 1 iff M_e halts on input w and halts with output 0 otherwise. In other words, $E \frown D$ would solve the halting problem. But we know, by **??**, that no such Turing machine can exist. □

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Bibliography