

## und.1 The Decision Problem is Unsolvable

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thm:decision-prob

**Theorem und.1.** *The decision problem is unsolvable.*

*Proof.* Suppose the decision problem were solvable, i.e., suppose there were a Turing machine  $D$  of the following sort. Whenever  $D$  is started on a tape that contains a **sentence**  $\psi$  of first-order logic as input,  $D$  eventually halts, and outputs 1 iff  $\psi$  is valid and 0 otherwise. Then we could solve the halting problem as follows. We construct a Turing machine  $E$  that, given as input the number  $e$  of Turing machine  $M_e$  and input  $w$ , computes the corresponding **sentence**  $\tau(M_e, w) \rightarrow \alpha(M_e, w)$  and halts, scanning the leftmost square on the tape. The machine  $E \frown D$  would then, given input  $e$  and  $w$ , first compute  $\tau(M_e, w) \rightarrow \alpha(M_e, w)$  and then run the decision problem machine  $D$  on that input.  $D$  halts with output 1 iff  $\tau(M_e, w) \rightarrow \alpha(M_e, w)$  is valid and outputs 0 otherwise. By ?? and ??,  $\tau(M_e, w) \rightarrow \alpha(M_e, w)$  is valid iff  $M_e$  halts on input  $w$ . Thus,  $E \frown D$ , given input  $e$  and  $w$  halts with output 1 iff  $M_e$  halts on input  $w$  and halts with output 0 otherwise. In other words,  $E \frown D$  would solve the halting problem. But we know, by ??, that no such Turing machine can exist.  $\square$

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## Bibliography