Theorem und.1. The decision problem is unsolvable.

Proof. Suppose the decision problem were solvable, i.e., suppose there were a Turing machine $D$ of the following sort. Whenever $D$ is started on a tape that contains a sentence $\psi$ of first-order logic as input, $D$ eventually halts, and outputs 1 iff $\psi$ is valid and 0 otherwise. Then we could solve the halting problem as follows. We construct a Turing machine $E$ that, given as input the number $e$ of Turing machine $M_e$ and input $w$, computes the corresponding sentence $\tau(M_e, w) \to \alpha(M_e, w)$ and halts, scanning the leftmost square on the tape. The machine $E \mathbin{\bowtie} D$ would then, given input $e$ and $w$, first compute $\tau(M_e, w) \to \alpha(M_e, w)$ and then run the decision problem machine $D$ on that input. $D$ halts with output 1 iff $\tau(M_e, w) \to \alpha(M_e, w)$ is valid and outputs 0 otherwise. By ?? and ??, $\tau(M_e, w) \to \alpha(M_e, w)$ is valid iff $M_e$ halts on input $w$. Thus, $E \mathbin{\bowtie} D$, given input $e$ and $w$ halts with output 1 iff $M_e$ halts on input $w$ and halts with output 0 otherwise. In other words, $E \mathbin{\bowtie} D$ would solve the halting problem. But we know, by ??, that no such Turing machine can exist.

Photo Credits

Bibliography