

und.1 The Halting Problem

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Assume we have fixed some finite descriptions of Turing machines. Using these, [explanation](#) we can enumerate Turing machines via their descriptions, say, ordered by the lexicographic ordering. Each Turing machine thus receives an *index*: its place in the enumeration M_1, M_2, M_3, \dots of Turing machine descriptions.

We know that there must be non-Turing-computable functions: the set of Turing machine descriptions—and hence the set of Turing machines—is enumerable, but the set of all functions from \mathbb{N} to \mathbb{N} is not. But we can find specific examples of non-computable function as well. One such function is the halting function.

Definition und.1 (Halting function). The *halting function* h is defined as

$$h(e, n) = \begin{cases} 0 & \text{if machine } M_e \text{ does not halt for input } n \\ 1 & \text{if machine } M_e \text{ halts for input } n \end{cases}$$

Definition und.2 (Halting problem). The *Halting Problem* is the problem of determining (for any e, n) whether the Turing machine M_e halts for an input of n strokes.

We show that h is not Turing-computable by showing that a related function, s , is not Turing-computable. This proof relies on the fact that anything that can be computed by a Turing machine can be computed using just two symbols: 0 and 1, and the fact that two Turing machines can be hooked together to create a single machine. [explanation](#)

Definition und.3. The function s is defined as

$$s(e) = \begin{cases} 0 & \text{if machine } M_e \text{ does not halt for input } e \\ 1 & \text{if machine } M_e \text{ halts for input } e \end{cases}$$

Lemma und.4. *The function s is not Turing computable.*

Proof. We suppose, for contradiction, that the function s is Turing-computable. Then there would be a Turing machine S that computes s . We may assume, without loss of generality, that when S halts, it does so while scanning the first square. This machine can be “hooked up” to another machine J , which halts if it is started on a blank tape (i.e., if it reads 0 in the initial state while scanning the square to the right of the end-of-tape symbol), and otherwise wanders off to the right, never halting. $S \frown J$, the machine created by hooking S to J , is a Turing machine, so it is M_e for some e (i.e., it appears somewhere in the enumeration). Start M_e on an input of e 1s. There are two possibilities: either M_e halts or it does not halt.

1. Suppose M_e halts for an input of e 1s. Then $s(e) = 1$. So S , when started on e , halts with a single 1 as output on the tape. Then J starts with a 1 on the tape. In that case J does not halt. But M_e is the machine $S \frown J$,

so it should do exactly what S followed by J would do. So M_e cannot halt for an input of e 1's.

2. Now suppose M_e does not halt for an input of e 1s. Then $s(e) = 0$, and S , when started on input e , halts with a blank tape. J , when started on a blank tape, immediately halts. Again, M_e does what S followed by J would do, so M_e must halt for an input of e 1's.

This shows there cannot be a Turing machine S : s is not Turing computable. □

Theorem und.5 (Unsolvability of the Halting Problem). *The halting problem is unsolvable, i.e., the function h is not Turing computable.* [tms:und:hal:](#)
[thm:halting-problem](#)

Proof. Suppose h were Turing computable, say, by a Turing machine H . We could use H to build a Turing machine that computes s : First, make a copy of the input (separated by a blank). Then move back to the beginning, and run H . We can clearly make a machine that does the former, and if H existed, we would be able to “hook it up” to such a modified doubling machine to get a new machine which would determine if M_e halts on input e , i.e., computes s . But we’ve already shown that no such machine can exist. Hence, h is also not Turing computable. □

Problem und.1. The Three Halting (3-Halt) problem is the problem of giving a decision procedure to determine whether or not an arbitrarily chosen Turing Machine halts for an input of three strokes on an otherwise blank tape. Prove that the 3-Halt problem is unsolvable.

Problem und.2. Show that if the halting problem is solvable for Turing machine and input pairs M_e and n where $e \neq n$, then it is also solvable for the cases where $e = n$.

Problem und.3. We proved that the halting problem is unsolvable if the input is a number e , which identifies a Turing machine M_e via an enumeration of all Turing machines. What if we allow the description of Turing machines from ?? directly as input? (This would require a larger alphabet of course.) Can there be a Turing machine which decides the halting problem but takes as input descriptions of Turing machines rather than indices? Explain why or why not.

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Bibliography