

und.1 The Halting Problem

tur:und:hal:sec Assume we have fixed some enumeration of Turing machine descriptions. Each explanation Turing machine thus receives an *index*: its place in the enumeration M_1, M_2, M_3, \dots of Turing machine descriptions.

We know that there must be non-Turing-computable functions: the set of Turing machine descriptions—and hence the set of Turing machines—is **enumerable**, but the set of all functions from \mathbb{N} to \mathbb{N} is not. But we can find specific examples of non-computable functions as well. One such function is the halting function.

Definition und.1 (Halting function). The *halting function* h is defined as

$$h(e, n) = \begin{cases} 0 & \text{if machine } M_e \text{ does not halt for input } n \\ 1 & \text{if machine } M_e \text{ halts for input } n \end{cases}$$

Definition und.2 (Halting problem). The *Halting Problem* is the problem of determining (for any e, n) whether the Turing machine M_e halts for an input of n strokes.

We show that h is not Turing-computable by showing that a related function, s , is not Turing-computable. This proof relies on the fact that anything that can be computed by a Turing machine can be computed by a disciplined Turing machine (??), and the fact that two Turing machines can be hooked together to create a single machine (??). explanation

Definition und.3. The function s is defined as

$$s(e) = \begin{cases} 0 & \text{if machine } M_e \text{ does not halt for input } e \\ 1 & \text{if machine } M_e \text{ halts for input } e \end{cases}$$

Lemma und.4. *The function s is not Turing computable.*

Proof. We suppose, for contradiction, that the function s is Turing computable. Then there would be a Turing machine S that computes s . We may assume, without loss of generality, that when S halts, it does so while scanning the first square (i.e., that it is disciplined). This machine can be “hooked up” to another machine J , which halts if it is started on input 0 (i.e., if it reads 0 in the initial state while scanning the square to the right of the end-of-tape symbol), and otherwise wanders off to the right, never halting. $S \frown J$, the machine created by hooking S to J , is a Turing machine, so it is M_e for some e (i.e., it appears somewhere in the enumeration). Start M_e on an input of e 1s. There are two possibilities: either M_e halts or it does not halt.

1. Suppose M_e halts for an input of e 1s. Then $s(e) = 1$. So S , when started on e , halts with a single 1 as output on the tape. Then J starts with a 1 on the tape. In that case J does not halt. But M_e is the machine $S \frown J$,

so it should do exactly what S followed by J would do (i.e., in this case, wander off to the right and never halt). So M_e cannot halt for an input of e 1's.

- Now suppose M_e does not halt for an input of e 1s. Then $s(e) = 0$, and S , when started on input e , halts with a blank tape. J , when started on a blank tape, immediately halts. Again, M_e does what S followed by J would do, so M_e must halt for an input of e 1's.

In each case we arrive at a contradiction with our assumption. This shows there cannot be a Turing machine S : s is not Turing computable. \square

Theorem und.5 (Unsolvability of the Halting Problem). *The halting problem is unsolvable, i.e., the function h is not Turing computable.*

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Proof. Suppose h were Turing computable, say, by a Turing machine H . We could use H to build a Turing machine that computes s : First, make a copy of the input (separated by a 0 symbol). Then move back to the beginning, and run H . We can clearly make a machine that does the former (see ??), and if H existed, we would be able to “hook it up” to such a copier machine to get a new machine which would determine if M_e halts on input e , i.e., computes s . But we’ve already shown that no such machine can exist. Hence, h is also not Turing computable. \square

Problem und.1. The Three Halting (3-Halt) problem is the problem of giving a decision procedure to determine whether or not an arbitrarily chosen Turing Machine halts for an input of three 1's on an otherwise blank tape. Prove that the 3-Halt problem is unsolvable.

Problem und.2. Show that if the halting problem is solvable for Turing machine and input pairs M_e and n where $e \neq n$, then it is also solvable for the cases where $e = n$.

Problem und.3. We proved that the halting problem is unsolvable if the input is a number e , which identifies a Turing machine M_e via an enumeration of all Turing machines. What if we allow the description of Turing machines from ?? directly as input? Can there be a Turing machine which decides the halting problem but takes as input descriptions of Turing machines rather than indices? Explain why or why not.

Problem und.4. Show that the *partial* function s' is defined as

$$s'(e) = \begin{cases} 1 & \text{if machine } M_e \text{ halts for input } e \\ \text{undefined} & \text{if machine } M_e \text{ does not halt for input } e \end{cases}$$

is Turing computable.

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Bibliography