und.1 The Decision Problem

We say that first-order logic is *decidable* iff there is an effective method for
determining whether or not a given *sentence* is valid. As it turns out, there
is no such method: the problem of deciding validity of first-order sentences is
unsolvable.

In order to establish this important negative result, we prove that the de-
cision problem cannot be solved by a Turing machine. That is, we show that
there is no Turing machine which, whenever it is started on a tape that contains
a first-order *sentence*, eventually halts and outputs either 1 or 0 depending on
whether the *sentence* is valid or not. By the Church–Turing thesis, every func-
tion which is computable is Turing computable. So if this “validity function”
were effectively computable at all, it would be Turing computable. If it isn’t
Turing computable, then, it also cannot be effectively computable.

Our strategy for proving that the decision problem is unsolvable is to reduce
the halting problem to it. This means the following: We have proved that the
function \( h(e, w) \) that halts with output 1 if the Turing machine described by \( e \)
halts on input \( w \) and outputs 0 otherwise, is not Turing computable. We will
show that if there were a Turing machine that decides validity of first-order
sentences, then there is also Turing machine that computes \( h \). Since \( h \) cannot
be computed by a Turing machine, there cannot be a Turing machine that
decides validity either.

The first step in this strategy is to show that for every input \( w \) and a Turing
machine \( M \), we can effectively describe a sentence \( \tau(M, w) \) representing the
instruction set of \( M \) and the input \( w \) and a sentence \( \alpha(M, w) \) expressing “\( M \)
eventually halts” such that:

\[
\vdash \tau(M, w) \rightarrow \alpha(M, w) \text{ iff } M \text{ halts for input } w.
\]

The bulk of our proof will consist in describing these sentences \( \tau(M, w) \) and \( \alpha(M, w) \)
and in verifying that \( \tau(M, w) \rightarrow \alpha(M, w) \) is valid iff \( M \) halts on input \( w \).

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Bibliography