siz.1 Reduction

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> This section proves non-enumerability by reduction, matching the results in **??**. An alternative, slightly more elaborate version matching the results in **??** is provided in **??**.

We proved that $\mathbb{B}^\omega$ is non-enumerable by a diagonalization argument. We used a similar diagonalization argument to show that $\wp(\mathbb{N})$ is non-enumerable. But here's another way we can prove that $\wp(\mathbb{N})$ is non-enumerable: show that *if* $\wp(\mathbb{N})$ *is enumerable then* $\mathbb{B}^\omega$ *is also enumerable*. Since we know $\mathbb{B}^\omega$ is non-enumerable, it will follow that $\wp(\mathbb{N})$ is too.

This is called *reducing* one problem to another. In this case, we reduce the problem of enumerating $\mathbb{B}^\omega$ to the problem of enumerating $\wp(\mathbb{N})$. A solution to the latter—an enumeration of $\wp(\mathbb{N})$—would yield a solution to the former—an enumeration of $\mathbb{B}^\omega$.

To reduce the problem of enumerating a set B to that of enumerating a set A, we provide a way of turning an enumeration of A into an enumeration of B. The easiest way to do that is to define a surjection $f\colon A \to B$. If x_1, x_2, … enumerates A, then $f(x_1)$, $f(x_2)$, … would enumerate B. In our case, we are looking for a surjection $f\colon \wp(\mathbb{N}) \to \mathbb{B}^\omega$.

Problem siz.1. Show that if there is an injective function $g\colon B \to A$, and B is non-enumerable, then so is A. Do this by showing how you can use g to turn an enumeration of A into one of B.

Proof of **??** *by reduction.* For reductio, suppose that $\wp(\mathbb{N})$ is enumerable, and thus that there is an enumeration of it, N_1, N_2, N_3, …

Define the function $f\colon \wp(\mathbb{N}) \to \mathbb{B}^\omega$ by letting $f(N)$ be the string s_k such that $s_k(n) = 1$ iff $n \in N$, and $s_k(n) = 0$ otherwise.

This clearly defines a function, since whenever $N \subseteq \mathbb{N}$, any $n \in \mathbb{N}$ either is an element of N or isn't. For instance, the set $2\mathbb{N} = \{2n : n \in \mathbb{N}\} = \{0, 2, 4, 6, \dots\}$ of even naturals gets mapped to the string $1010101\dots$; $\emptyset$ gets mapped to $0000\dots$; $\mathbb{N}$ gets mapped to $1111\dots$.

It is also surjective: every string of 0s and 1s corresponds to some set of natural numbers, namely the one which has as its members those natural numbers corresponding to the places where the string has 1s. More precisely, if $s \in \mathbb{B}^\omega$, then define $N \subseteq \mathbb{N}$ by:

$$N = \{n \in \mathbb{N} : s(n) = 1\}$$

Then $f(N) = s$, as can be verified by consulting the definition of f.

Now consider the list

$$f(N_1), f(N_2), f(N_3), \dots$$

Since f is surjective, every member of $\mathbb{B}^\omega$ must appear as a value of f for some argument, and so must appear on the list. This list must therefore enumerate all of $\mathbb{B}^\omega$.

So if $\wp(\mathbb{N})$ were enumerable, $\mathbb{B}^\omega$ would be enumerable. But $\mathbb{B}^\omega$ is non-enumerable (**??**). Hence $\wp(\mathbb{N})$ is non-enumerable. □

Problem siz.2. Show that the set of all *sets of* pairs of natural numbers, i.e., $\wp(\mathbb{N} \times \mathbb{N})$, is non-enumerable by a reduction argument.

Problem siz.3. Show that $\mathbb{N}^\omega$, the set of infinite sequences of natural numbers, is non-enumerable by a reduction argument.

Problem siz.4. Let S be the set of all surjections from $\mathbb{N}$ to the set $\{0, 1\}$, i.e., S consists of all surjections $f : \mathbb{N} \to \mathbb{B}$. Show that S is non-enumerable.

Problem siz.5. Show that the set $\mathbb{R}$ of all real numbers is non-enumerable.

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Bibliography