

## siz.1 Pairing Functions and Codes

sfr:siz:pai:  
sec Cantor's zig-zag method makes the enumerability of  $\mathbb{N}^n$  visually evident. But explanation let us focus on our array depicting  $\mathbb{N}^2$ . Following the zig-zag line in the array and counting the places, we can check that  $\langle 1, 2 \rangle$  is associated with the number 7. However, it would be nice if we could compute this more directly. That is, it would be nice to have to hand the *inverse* of the zig-zag enumeration,  $g: \mathbb{N}^2 \rightarrow \mathbb{N}$ , such that

$$g(\langle 0, 0 \rangle) = 0, g(\langle 0, 1 \rangle) = 1, g(\langle 1, 0 \rangle) = 2, \dots, g(\langle 1, 2 \rangle) = 7, \dots$$

This would enable us to calculate exactly where  $\langle n, m \rangle$  will occur in our enumeration.

In fact, we can define  $g$  directly by making two observations. First: if the  $n$ th row and  $m$ th column contains value  $v$ , then the  $(n+1)$ st row and  $(m-1)$ st column contains value  $v+1$ . Second: the first row of our enumeration consists of the triangular numbers, starting with 0, 1, 3, 6, etc. The  $k$ th triangular number is the sum of the natural numbers  $< k$ , which can be computed as  $k(k+1)/2$ . Putting these two observations together, consider this function:

$$g(n, m) = \frac{(n+m+1)(n+m)}{2} + n$$

We often just write  $g(n, m)$  rather than  $g(\langle n, m \rangle)$ , since it is easier on the eyes. This tells you first to determine the  $(n+m)$ <sup>th</sup> triangle number, and then add  $n$  to it. And it populates the array in exactly the way we would like. So in particular, the pair  $\langle 1, 2 \rangle$  is sent to  $\frac{4 \times 3}{2} + 1 = 7$ .

This function  $g$  is the *inverse* of an enumeration of a set of pairs. Such functions are called *pairing functions*.

**Definition siz.1 (Pairing function).** A function  $f: A \times B \rightarrow \mathbb{N}$  is an arithmetical *pairing function* if  $f$  is injective. We also say that  $f$  *encodes*  $A \times B$ , and that  $f(x, y)$  is the *code* for  $\langle x, y \rangle$ .

We can use pairing functions to encode, e.g., pairs of natural numbers; or, in other words, we can represent each *pair* of elements using a *single* number. Using the inverse of the pairing function, we can *decode* the number, i.e., find out which pair it represents. explanation

**Problem siz.1.** Give an enumeration of the set of all non-negative rational numbers.

**Problem siz.2.** Show that  $\mathbb{Q}$  is **enumerable**. Recall that any rational number can be written as a fraction  $z/m$  with  $z \in \mathbb{Z}$ ,  $m \in \mathbb{N}^+$ .

**Problem siz.3.** Define an enumeration of  $\mathbb{B}^*$ .

**Problem siz.4.** Recall from your introductory logic course that each possible truth table expresses a truth function. In other words, the truth functions are all functions from  $\mathbb{B}^k \rightarrow \mathbb{B}$  for some  $k$ . Prove that the set of all truth functions is enumerable.

**Problem siz.5.** Show that the set of all finite subsets of an arbitrary infinite enumerable set is enumerable.

**Problem siz.6.** A subset of  $\mathbb{N}$  is said to be *cofinite* iff it is the complement of a finite set  $\mathbb{N}$ ; that is,  $A \subseteq \mathbb{N}$  is cofinite iff  $\mathbb{N} \setminus A$  is finite. Let  $I$  be the set whose elements are exactly the finite and cofinite subsets of  $\mathbb{N}$ . Show that  $I$  is enumerable.

**Problem siz.7.** Show that the enumerable union of enumerable sets is enumerable. That is, whenever  $A_1, A_2, \dots$  are sets, and each  $A_i$  is enumerable, then the union  $\bigcup_{i=1}^{\infty} A_i$  of all of them is also enumerable. [NB: this is hard!]

**Problem siz.8.** Let  $f: A \times B \rightarrow \mathbb{N}$  be an arbitrary pairing function. Show that the inverse of  $f$  is an enumeration of  $A \times B$ .

**Problem siz.9.** Specify a function that encodes  $\mathbb{N}^3$ .

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## Bibliography