

siz.1 Pairing Functions and Codes

sfr:siz:pai:
sec Cantor's zig-zag method makes the enumerability of \mathbb{N}^n visually evident. But explanation let us focus on our array depicting \mathbb{N}^2 . Following the zig-zag line in the array and counting the places, we can check that $\langle 1, 2 \rangle$ is associated with the number 7. However, it would be nice if we could compute this more directly. That is, it would be nice to have to hand the *inverse* of the zig-zag enumeration, $g: \mathbb{N}^2 \rightarrow \mathbb{N}$, such that

$$g(\langle 0, 0 \rangle) = 0, g(\langle 0, 1 \rangle) = 1, g(\langle 1, 0 \rangle) = 2, \dots, g(\langle 1, 2 \rangle) = 7, \dots$$

This would enable us to calculate exactly where $\langle n, m \rangle$ will occur in our enumeration.

In fact, we can define g directly by making two observations. First: if the n th row and m th column contains value v , then the $(n+1)$ st row and $(m-1)$ st column contains value $v+1$. Second: the first row of our enumeration consists of the triangular numbers, starting with 0, 1, 3, 6, etc. The k th triangular number is the sum of the natural numbers $< k$, which can be computed as $k(k+1)/2$. Putting these two observations together, consider this function:

$$g(n, m) = \frac{(n+m+1)(n+m)}{2} + n$$

We often just write $g(n, m)$ rather than $g(\langle n, m \rangle)$, since it is easier on the eyes. This tells you first to determine the $(n+m)$ th triangle number, and then add n to it. And it populates the array in exactly the way we would like. So in particular, the pair $\langle 1, 2 \rangle$ is sent to $\frac{4 \times 3}{2} + 1 = 7$.

This function g is the *inverse* of an enumeration of a set of pairs. Such functions are called *pairing functions*.

Definition siz.1 (Pairing function). A function $f: A \times B \rightarrow \mathbb{N}$ is an arithmetical *pairing function* if f is injective. We also say that f *encodes* $A \times B$, and that $f(x, y)$ is the *code* for $\langle x, y \rangle$.

We can use pairing functions to encode, e.g., pairs of natural numbers; or, in other words, we can represent each *pair* of elements using a *single* number. Using the inverse of the pairing function, we can *decode* the number, i.e., find out which pair it represents. explanation

Problem siz.1. Give an enumeration of the set of all non-negative rational numbers.

Problem siz.2. Show that \mathbb{Q} is **enumerable**. Recall that any rational number can be written as a fraction z/m with $z \in \mathbb{Z}$, $m \in \mathbb{N}^+$.

Problem siz.3. Define an enumeration of \mathbb{B}^* .

Problem siz.4. Recall from your introductory logic course that each possible truth table expresses a truth function. In other words, the truth functions are all functions from $\mathbb{B}^k \rightarrow \mathbb{B}$ for some k . Prove that the set of all truth functions is enumerable.

Problem siz.5. Show that the set of all finite subsets of an arbitrary infinite enumerable set is enumerable.

Problem siz.6. A subset of \mathbb{N} is said to be *cofinite* iff it is the complement of a finite set \mathbb{N} ; that is, $A \subseteq \mathbb{N}$ is cofinite iff $\mathbb{N} \setminus A$ is finite. Let I be the set whose elements are exactly the finite and cofinite subsets of \mathbb{N} . Show that I is enumerable.

Problem siz.7. Show that the enumerable union of enumerable sets is enumerable. That is, whenever A_1, A_2, \dots are sets, and each A_i is enumerable, then the union $\bigcup_{i=1}^{\infty} A_i$ of all of them is also enumerable. [NB: this is hard!]

Problem siz.8. Let $f: A \times B \rightarrow \mathbb{N}$ be an arbitrary pairing function. Show that the inverse of f is an enumeration of $A \times B$.

Problem siz.9. Specify a function that encodes \mathbb{N}^3 .

Photo Credits

Bibliography