

siz.1 Non-enumerable Sets

sfr:siz:nen:
sec

This section proves the non-enumerability of \mathbb{B}^ω and $\wp(\mathbb{Z}^+)$ using the definition in ???. It is designed to be a little more elementary and a little more detailed than the version in ???

Some sets, such as the set \mathbb{Z}^+ of positive integers, are infinite. So far we've seen examples of infinite sets which were all **enumerable**. However, there are also infinite sets which do not have this property. Such sets are called *non-enumerable*.

First of all, it is perhaps already surprising that there are **non-enumerable** sets. For any **enumerable** set A there is a **surjective** function $f: \mathbb{Z}^+ \rightarrow A$. If a set is **non-enumerable** there is no such function. That is, no function mapping the infinitely many **elements** of \mathbb{Z}^+ to A can exhaust all of A . So there are "more" **elements** of A than the infinitely many positive integers.

How would one prove that a set is **non-enumerable**? You have to show that no such surjective function can exist. Equivalently, you have to show that the elements of A cannot be enumerated in a one way infinite list. The best way to do this is to show that every list of **elements** of A must leave at least one element out; or that no function $f: \mathbb{Z}^+ \rightarrow A$ can be **surjective**. We can do this using Cantor's *diagonal method*. Given a list of **elements** of A , say, x_1, x_2, \dots , we construct another element of A which, by its construction, cannot possibly be on that list.

Our first example is the set \mathbb{B}^ω of all infinite, non-gappy sequences of 0's and 1's.

sfr:siz:nen:
thm:nonenum-bin-omega

Theorem siz.1. \mathbb{B}^ω is *non-enumerable*.

Proof. Suppose, by way of contradiction, that \mathbb{B}^ω is **enumerable**, i.e., suppose that there is a list $s_1, s_2, s_3, s_4, \dots$ of all **elements** of \mathbb{B}^ω . Each of these s_i is itself an infinite sequence of 0's and 1's. Let's call the j -th element of the i -th sequence in this list $s_i(j)$. Then the i -th sequence s_i is

$$s_i(1), s_i(2), s_i(3), \dots$$

We may arrange this list, and the elements of each sequence s_i in it, in an array:

	1	2	3	4	...
1	$s_1(1)$	$s_1(2)$	$s_1(3)$	$s_1(4)$...
2	$s_2(1)$	$s_2(2)$	$s_2(3)$	$s_2(4)$...
3	$s_3(1)$	$s_3(2)$	$s_3(3)$	$s_3(4)$...
4	$s_4(1)$	$s_4(2)$	$s_4(3)$	$s_4(4)$...
\vdots	\vdots	\vdots	\vdots	\vdots	\ddots

The labels down the side give the number of the sequence in the list s_1, s_2, \dots ; the numbers across the top label the **elements** of the individual sequences. For instance, $s_1(1)$ is a name for whatever number, a 0 or a 1, is the first **element** in the sequence s_1 , and so on.

Now we construct an infinite sequence, \bar{s} , of 0's and 1's which cannot possibly be on this list. The definition of \bar{s} will depend on the list s_1, s_2, \dots . Any infinite list of infinite sequences of 0's and 1's gives rise to an infinite sequence \bar{s} which is guaranteed to not appear on the list.

To define \bar{s} , we specify what all its **elements** are, i.e., we specify $\bar{s}(n)$ for all $n \in \mathbb{Z}^+$. We do this by reading down the diagonal of the array above (hence the name “diagonal method”) and then changing every 1 to a 0 and every 0 to a 1. More abstractly, we define $\bar{s}(n)$ to be 0 or 1 according to whether the n -th **element** of the diagonal, $s_n(n)$, is 1 or 0.

$$\bar{s}(n) = \begin{cases} 1 & \text{if } s_n(n) = 0 \\ 0 & \text{if } s_n(n) = 1. \end{cases}$$

If you like formulas better than definitions by cases, you could also define $\bar{s}(n) = 1 - s_n(n)$.

Clearly \bar{s} is an infinite sequence of 0's and 1's, since it is just the mirror sequence to the sequence of 0's and 1's that appear on the diagonal of our array. So \bar{s} is an **element** of \mathbb{B}^ω . But it cannot be on the list s_1, s_2, \dots . Why not?

It can't be the first sequence in the list, s_1 , because it differs from s_1 in the first **element**. Whatever $s_1(1)$ is, we defined $\bar{s}(1)$ to be the opposite. It can't be the second sequence in the list, because \bar{s} differs from s_2 in the second element: if $s_2(2)$ is 0, $\bar{s}(2)$ is 1, and vice versa. And so on.

More precisely: if \bar{s} were on the list, there would be some k so that $\bar{s} = s_k$. Two sequences are identical iff they agree at every place, i.e., for any n , $\bar{s}(n) = s_k(n)$. So in particular, taking $n = k$ as a special case, $\bar{s}(k) = s_k(k)$ would have to hold. $s_k(k)$ is either 0 or 1. If it is 0 then $\bar{s}(k)$ must be 1—that's how we defined \bar{s} . But if $s_k(k) = 1$ then, again because of the way we defined \bar{s} , $\bar{s}(k) = 0$. In either case $\bar{s}(k) \neq s_k(k)$.

We started by assuming that there is a list of **elements** of \mathbb{B}^ω , s_1, s_2, \dots . From this list we constructed a sequence \bar{s} which we proved cannot be on the list. But it definitely is a sequence of 0's and 1's if all the s_i are sequences of 0's and 1's, i.e., $\bar{s} \in \mathbb{B}^\omega$. This shows in particular that there can be no list of *all elements* of \mathbb{B}^ω , since for any such list we could also construct a sequence \bar{s} guaranteed to not be on the list, so the assumption that there is a list of all sequences in \mathbb{B}^ω leads to a contradiction. \square

explanation

This proof method is called “diagonalization” because it uses the diagonal of the array to define \bar{s} . Diagonalization need not involve the presence of an array: we can show that sets are not **enumerable** by using a similar idea even when no array and no actual diagonal is involved.

Theorem siz.2. $\wp(\mathbb{Z}^+)$ is not **enumerable**.

*sfr:siz:nen:
thm:nonenum-pownat*

Proof. We proceed in the same way, by showing that for every list of subsets of \mathbb{Z}^+ there is a subset of \mathbb{Z}^+ which cannot be on the list. Suppose the following is a given list of subsets of \mathbb{Z}^+ :

$$Z_1, Z_2, Z_3, \dots$$

We now define a set \bar{Z} such that for any $n \in \mathbb{Z}^+$, $n \in \bar{Z}$ iff $n \notin Z_n$:

$$\bar{Z} = \{n \in \mathbb{Z}^+ : n \notin Z_n\} \quad \square$$

\bar{Z} is clearly a set of positive integers, since by assumption each Z_n is, and thus $\bar{Z} \in \wp(\mathbb{Z}^+)$. But \bar{Z} cannot be on the list. To show this, we'll establish that for each $k \in \mathbb{Z}^+$, $\bar{Z} \neq Z_k$.

So let $k \in \mathbb{Z}^+$ be arbitrary. We've defined \bar{Z} so that for any $n \in \mathbb{Z}^+$, $n \in \bar{Z}$ iff $n \notin Z_n$. In particular, taking $n = k$, $k \in \bar{Z}$ iff $k \notin Z_k$. But this shows that $\bar{Z} \neq Z_k$, since k is an element of one but not the other, and so \bar{Z} and Z_k have different elements. Since k was arbitrary, \bar{Z} is not on the list Z_1, Z_2, \dots

The preceding proof did not mention a diagonal, but you can think of it as involving a diagonal if you picture it this way: explanation Imagine the sets Z_1, Z_2, \dots , written in an array, where each element $j \in Z_i$ is listed in the j -th column. Say the first four sets on that list are $\{1, 2, 3, \dots\}$, $\{2, 4, 6, \dots\}$, $\{1, 2, 5\}$, and $\{3, 4, 5, \dots\}$. Then the array would begin with

$$\begin{array}{cccccccc} Z_1 = \{ & \mathbf{1}, & 2, & 3, & 4, & 5, & 6, & \dots \} \\ Z_2 = \{ & & \mathbf{2}, & & 4, & & 6, & \dots \} \\ Z_3 = \{ & 1, & 2, & & & 5, & & \} \\ Z_4 = \{ & & & 3, & \mathbf{4}, & 5, & 6, & \dots \} \\ & \vdots & & & & \ddots & & \end{array}$$

Then \bar{Z} is the set obtained by going down the diagonal, leaving out any numbers that appear along the diagonal and include those j where the array has a gap in the j -th row/column. In the above case, we would leave out 1 and 2, include 3, leave out 4, etc.

Problem siz.1. Show that $\wp(\mathbb{N})$ is non-enumerable by a diagonal argument.

Problem siz.2. Show that the set of functions $f: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ is non-enumerable by an explicit diagonal argument. That is, show that if f_1, f_2, \dots , is a list of functions and each $f_i: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$, then there is some $\bar{f}: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ not on this list.

Photo Credits

Bibliography