

## siz.1 Non-enumerable Sets

sfr:siz:nen:  
sec

Some sets, such as the set  $\mathbb{Z}^+$  of positive integers, are infinite. So far we've seen examples of infinite sets which were all **enumerable**. However, there are also infinite sets which do not have this property. Such sets are called **non-enumerable**.

First of all, it is perhaps already surprising that there are **non-enumerable** sets. For any **enumerable** set  $X$  there is a **surjective** function  $f: \mathbb{Z}^+ \rightarrow X$ . If a set is **non-enumerable** there is no such function. That is, no function mapping the infinitely many **elements** of  $\mathbb{Z}^+$  to  $X$  can exhaust all of  $X$ . So there are "more" **elements** of  $X$  than the infinitely many positive integers.

How would one prove that a set is **non-enumerable**? You have to show that no such surjective function can exist. Equivalently, you have to show that the elements of  $X$  cannot be enumerated in a one way infinite list. The best way to do this is to show that every list of **elements** of  $X$  must leave at least one element out; or that no function  $f: \mathbb{Z}^+ \rightarrow X$  can be surjective. We can do this using Cantor's *diagonal method*. Given a list of **elements** of  $X$ , say,  $x_1, x_2, \dots$ , we construct another element of  $X$  which, by its construction, cannot possibly be on that list.

Our first example is the set  $\mathbb{B}^\omega$  of all infinite, non-gappy sequences of 0's and 1's.

sfr:siz:nen:  
thm-nonenum-bin-omega

**Theorem siz.1.**  $\mathbb{B}^\omega$  is **non-enumerable**.

*Proof.* Suppose, by way of contradiction, that  $\mathbb{B}^\omega$  is **enumerable**, i.e., suppose that there is a list  $s_1, s_2, s_3, s_4, \dots$  of all **elements** of  $\mathbb{B}^\omega$ . Each of these  $s_i$  is itself an infinite sequence of 0's and 1's. Let's call the  $j$ -th element of the  $i$ -th sequence in this list  $s_i(j)$ . Then the  $i$ -th sequence  $s_i$  is

$$s_i(1), s_i(2), s_i(3), \dots$$

We may arrange this list, and the elements of each sequence  $s_i$  in it, in an array:

	1	2	3	4	...
1	$\mathbf{s_1(1)}$	$s_1(2)$	$s_1(3)$	$s_1(4)$	...
2	$s_2(1)$	$\mathbf{s_2(2)}$	$s_2(3)$	$s_2(4)$	...
3	$s_3(1)$	$s_3(2)$	$\mathbf{s_3(3)}$	$s_3(4)$	...
4	$s_4(1)$	$s_4(2)$	$s_4(3)$	$\mathbf{s_4(4)}$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The labels down the side give the number of the sequence in the list  $s_1, s_2, \dots$ ; the numbers across the top label the **elements** of the individual sequences. For instance,  $s_1(1)$  is a name for whatever number, a 0 or a 1, is the first **element** in the sequence  $s_1$ , and so on.

Now we construct an infinite sequence,  $\bar{s}$ , of 0's and 1's which cannot possibly be on this list. The definition of  $\bar{s}$  will depend on the list  $s_1, s_2, \dots$ . Any

infinite list of infinite sequences of 0's and 1's gives rise to an infinite sequence  $\bar{s}$  which is guaranteed to not appear on the list.

To define  $\bar{s}$ , we specify what all its **elements** are, i.e., we specify  $\bar{s}(n)$  for all  $n \in \mathbb{Z}^+$ . We do this by reading down the diagonal of the array above (hence the name “diagonal method”) and then changing every 1 to a 0 and every 0 to a 1. More abstractly, we define  $\bar{s}(n)$  to be 0 or 1 according to whether the  $n$ -th **element** of the diagonal,  $s_n(n)$ , is 1 or 0.

$$\bar{s}(n) = \begin{cases} 1 & \text{if } s_n(n) = 0 \\ 0 & \text{if } s_n(n) = 1. \end{cases}$$

If you like formulas better than definitions by cases, you could also define  $\bar{s}(n) = 1 - s_n(n)$ .

Clearly  $\bar{s}$  is a non-gappy infinite sequence of 0's and 1's, since it is just the mirror sequence to the sequence of 0's and 1's that appear on the diagonal of our array. So  $\bar{s}$  is an **element** of  $\mathbb{B}^\omega$ . But it cannot be on the list  $s_1, s_2, \dots$ . Why not?

It can't be the first sequence in the list,  $s_1$ , because it differs from  $s_1$  in the first **element**. Whatever  $s_1(1)$  is, we defined  $\bar{s}(1)$  to be the opposite. It can't be the second sequence in the list, because  $\bar{s}$  differs from  $s_2$  in the second element: if  $s_2(2)$  is 0,  $\bar{s}(2)$  is 1, and vice versa. And so on.

More precisely: if  $\bar{s}$  were on the list, there would be some  $k$  so that  $\bar{s} = s_k$ . Two sequences are identical iff they agree at every place, i.e., for any  $n$ ,  $\bar{s}(n) = s_k(n)$ . So in particular, taking  $n = k$  as a special case,  $\bar{s}(k) = s_k(k)$  would have to hold.  $s_k(k)$  is either 0 or 1. If it is 0 then  $\bar{s}(k)$  must be 1—that's how we defined  $\bar{s}$ . But if  $s_k(k) = 1$  then, again because of the way we defined  $\bar{s}$ ,  $\bar{s}(k) = 0$ . In either case  $\bar{s}(k) \neq s_k(k)$ .

We started by assuming that there is a list of **elements** of  $\mathbb{B}^\omega$ ,  $s_1, s_2, \dots$ . From this list we constructed a sequence  $\bar{s}$  which we proved cannot be on the list. But it definitely is a sequence of 0's and 1's if all the  $s_i$  are sequences of 0's and 1's, i.e.,  $\bar{s} \in \mathbb{B}^\omega$ . This shows in particular that there can be no list of *all* **elements** of  $\mathbb{B}^\omega$ , since for any such list we could also construct a sequence  $\bar{s}$  guaranteed to not be on the list, so the assumption that there is a list of all sequences in  $\mathbb{B}^\omega$  leads to a contradiction.  $\square$

explanation

This proof method is called “diagonalization” because it uses the diagonal of the array to define  $\bar{s}$ . Diagonalization need not involve the presence of an array: we can show that sets are not **enumerable** by using a similar idea even when no array and no actual diagonal is involved.

**Theorem siz.2.**  $\wp(\mathbb{Z}^+)$  is not **enumerable**.

*sfr:siz:nen:  
thm-nonenum-pownat*

*Proof.* We proceed in the same way, by showing that for every list of subsets of  $\mathbb{Z}^+$  there is a subset of  $\mathbb{Z}^+$  which cannot be on the list. Suppose the following is a given list of subsets of  $\mathbb{Z}^+$ :

$$Z_1, Z_2, Z_3, \dots$$

We now define a set  $\bar{Z}$  such that for any  $n \in \mathbb{Z}^+$ ,  $n \in \bar{Z}$  iff  $n \notin Z_n$ :

$$\bar{Z} = \{n \in \mathbb{Z}^+ : n \notin Z_n\}$$

$\bar{Z}$  is clearly a set of positive integers, since by assumption each  $Z_n$  is, and thus  $\bar{Z} \in \wp(\mathbb{Z}^+)$ . But  $\bar{Z}$  cannot be on the list. To show this, we'll establish that for each  $k \in \mathbb{Z}^+$ ,  $\bar{Z} \neq Z_k$ .

So let  $k \in \mathbb{Z}^+$  be arbitrary. We've defined  $\bar{Z}$  so that for any  $n \in \mathbb{Z}^+$ ,  $n \in \bar{Z}$  iff  $n \notin Z_n$ . In particular, taking  $n = k$ ,  $k \in \bar{Z}$  iff  $k \notin Z_k$ . But this shows that  $\bar{Z} \neq Z_k$ , since  $k$  is an element of one but not the other, and so  $\bar{Z}$  and  $Z_k$  have different elements. Since  $k$  was arbitrary,  $\bar{Z}$  is not on the list  $Z_1, Z_2, \dots$   $\square$

The preceding proof did not mention a diagonal, but you can think of it as involving a diagonal if you picture it this way: [explanation](#) Imagine the sets  $Z_1, Z_2, \dots$ , written in an array, where each element  $j \in Z_i$  is listed in the  $j$ -th column. Say the first four sets on that list are  $\{1, 2, 3, \dots\}$ ,  $\{2, 4, 6, \dots\}$ ,  $\{1, 2, 5\}$ , and  $\{3, 4, 5, \dots\}$ . Then the array would begin with

$$\begin{array}{l} Z_1 = \{ \mathbf{1}, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad \dots \} \\ Z_2 = \{ \quad \mathbf{2}, \quad \quad 4, \quad \quad 6, \quad \dots \} \\ Z_3 = \{ 1, \quad 2, \quad \quad \quad 5 \quad \quad \quad \} \\ Z_4 = \{ \quad \quad \quad 3, \quad \mathbf{4}, \quad 5, \quad 6, \quad \dots \} \\ \quad \quad \quad \vdots \quad \quad \quad \ddots \end{array}$$

Then  $\bar{Z}$  is the set obtained by going down the diagonal, leaving out any numbers that appear along the diagonal and include those  $j$  where the array has a gap in the  $j$ -th row/column. In the above case, we would leave out 1 and 2, include 3, leave out 4, etc.

**Problem siz.1.** Show that  $\wp(\mathbb{N})$  is non-enumerable by a diagonal argument.

**Problem siz.2.** Show that the set of functions  $f: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$  is non-enumerable by an explicit diagonal argument. That is, show that if  $f_1, f_2, \dots$ , is a list of functions and each  $f_i: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ , then there is some  $\bar{f}: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$  not on this list.

## Photo Credits

## Bibliography