siz.1 Non-enumerable Sets

sfr:siz:nen:sec Some sets, such as the set $\mathbb{Z}^+$ of positive integers, are infinite. So far we've seen examples of infinite sets which were all enumerable. However, there are also infinite sets which do not have this property. Such sets are called *non-enumerable*.

First of all, it is perhaps already surprising that there are non-enumerable sets. For any enumerable set X there is a surjective function $f\colon \mathbb{Z}^+ \to X$. If a set is non-enumerable there is no such function. That is, no function mapping the infinitely many elements of $\mathbb{Z}^+$ to X can exhaust all of X. So there are "more" elements of X than the infinitely many positive integers.

How would one prove that a set is non-enumerable? You have to show that no such surjective function can exist. Equivalently, you have to show that the elements of X cannot be enumerated in a one way infinite list. The best way to do this is to show that every list of elements of X must leave at least one element out; or that no function $f\colon \mathbb{Z}^+ \to X$ can be surjective. We can do this using Cantor's *diagonal method*. Given a list of elements of X, say, x_1, x_2, ..., we construct another element of X which, by its construction, cannot possibly be on that list.

Our first example is the set $\mathbb{B}^\omega$ of all infinite, non-gappy sequences of 0's and 1's.

sfr:siz:nen:thm-nonenum-bin-omega **Theorem siz.1.** *$\mathbb{B}^\omega$ is non-enumerable.*

Proof. Suppose, by way of contradiction, that $\mathbb{B}^\omega$ is enumerable, i.e., suppose that there is a list s_1, s_2, s_3, s_4, ... of all elements of $\mathbb{B}^\omega$. Each of these s_i is itself an infinite sequence of 0's and 1's. Let's call the j-th element of the i-th sequence in this list $s_i(j)$. Then the i-th sequence s_i is

$$s_i(1), s_i(2), s_i(3), \ldots$$

We may arrange this list, and the elements of each sequence s_i in it, in an array:

	1	2	3	4	...
1	$\mathbf{s_1(1)}$	$s_1(2)$	$s_1(3)$	$s_1(4)$	...
2	$s_2(1)$	$\mathbf{s_2(2)}$	$s_2(3)$	$s_2(4)$	...
3	$s_3(1)$	$s_3(2)$	$\mathbf{s_3(3)}$	$s_3(4)$	...
4	$s_4(1)$	$s_4(2)$	$s_4(3)$	$\mathbf{s_4(4)}$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The labels down the side give the number of the sequence in the list s_1, s_2, ...; the numbers across the top label the elements of the individual sequences. For instance, $s_1(1)$ is a name for whatever number, a 0 or a 1, is the first element in the sequence s_1, and so on.

Now we construct an infinite sequence, $\overline{s}$, of 0's and 1's which cannot possibly be on this list. The definition of $\overline{s}$ will depend on the list s_1, s_2, ... Any infinite list of infinite sequences of 0's and 1's gives rise to an infinite sequence $\overline{s}$ which is guaranteed to not appear on the list.

To define $\overline{s}$, we specify what all its elements are, i.e., we specify $\overline{s}(n)$ for all $n \in \mathbb{Z}^+$. We do this by reading down the diagonal of the array above (hence the name "diagonal method") and then changing every 1 to a 0 and every 1 to a 0. More abstractly, we define $\overline{s}(n)$ to be 0 or 1 according to whether the n-th element of the diagonal, $s_n(n)$, is 1 or 0.

$$\overline{s}(n) = \begin{cases} 1 & \text{if } s_n(n) = 0 \\ 0 & \text{if } s_n(n) = 1. \end{cases}$$

If you like formulas better than definitions by cases, you could also define $\overline{s}(n) = 1 - s_n(n)$.

Clearly $\overline{s}$ is a non-gappy infinite sequence of 0's and 1's, since it is just the mirror sequence to the sequence of 0's and 1's that appear on the diagonal of our array. So $\overline{s}$ is an element of $\mathbb{B}^\omega$. But it cannot be on the list s_1, s_2, ... Why not?

It can't be the first sequence in the list, s_1, because it differs from s_1 in the first element. Whatever $s_1(1)$ is, we defined $\overline{s}(1)$ to be the opposite. It can't be the second sequence in the list, because $\overline{s}$ differs from s_2 in the second element: if $s_2(2)$ is 0, $\overline{s}(2)$ is 1, and vice versa. And so on.

More precisely: if $\overline{s}$ were on the list, there would be some k so that $\overline{s} = s_k$. Two sequences are identical iff they agree at every place, i.e., for any n, $\overline{s}(n) = s_k(n)$. So in particular, taking $n = k$ as a special case, $\overline{s}(k) = s_k(k)$ would have to hold. $s_k(k)$ is either 0 or 1. If it is 0 then $\overline{s}(k)$ must be 1—that's how we defined $\overline{s}$. But if $s_k(k) = 1$ then, again because of the way we defined $\overline{s}$, $\overline{s}(k) = 0$. In either case $\overline{s}(k) \neq s_k(k)$.

We started by assuming that there is a list of elements of $\mathbb{B}^\omega$, s_1, s_2, ... From this list we constructed a sequence $\overline{s}$ which we proved cannot be on the list. But it definitely is a sequence of 0's and 1's if all the s_i are sequences of 0's and 1's, i.e., $\overline{s} \in \mathbb{B}^\omega$. This shows in particular that there can be no list of *all* elements of $\mathbb{B}^\omega$, since for any such list we could also construct a sequence $\overline{s}$ guaranteed to not be on the list, so the assumption that there is a list of all sequences in $\mathbb{B}^\omega$ leads to a contradiction. □

explanation This proof method is called "diagonalization" because it uses the diagonal of the array to define $\overline{s}$. Diagonalization need not involve the presence of an array: we can show that sets are not enumerable by using a similar idea even when no array and no actual diagonal is involved.

Theorem siz.2. $\wp(\mathbb{Z}^+)$ *is not enumerable.* *sfr:siz:nen:thm-nonenum-pownat*

Proof. We proceed in the same way, by showing that for every list of subsets of $\mathbb{Z}^+$ there is a subset of $\mathbb{Z}^+$ which cannot be on the list. Suppose the following is a given list of subsets of $\mathbb{Z}^+$:

$$Z_1, Z_2, Z_3, \ldots$$

CC–BY

We now define a set $\overline{Z}$ such that for any $n \in \mathbb{Z}^+$, $n \in \overline{Z}$ iff $n \notin Z_n$:

$$\overline{Z} = \{n \in \mathbb{Z}^+ : n \notin Z_n\}$$

$\overline{Z}$ is clearly a set of positive integers, since by assumption each Z_n is, and thus $\overline{Z} \in \wp(\mathbb{Z}^+)$. But $\overline{Z}$ cannot be on the list. To show this, we'll establish that for each $k \in \mathbb{Z}^+$, $\overline{Z} \neq Z_k$.

So let $k \in \mathbb{Z}^+$ be arbitrary. We've defined $\overline{Z}$ so that for any $n \in \mathbb{Z}^+$, $n \in \overline{Z}$ iff $n \notin Z_n$. In particular, taking $n = k$, $k \in \overline{Z}$ iff $k \notin Z_k$. But this shows that $\overline{Z} \neq Z_k$, since k is an element of one but not the other, and so $\overline{Z}$ and Z_k have different elements. Since k was arbitrary, $\overline{Z}$ is not on the list Z_1, Z_2, … □

The preceding proof did not mention a diagonal, but you can think of it as involving a diagonal if you picture it this way: Imagine the sets Z_1, Z_2, …, written in an array, where each element $j \in Z_i$ is listed in the j-th column. Say the first four sets on that list are $\{1, 2, 3, \dots\}$, $\{2, 4, 6, \dots\}$, $\{1, 2, 5\}$, and $\{3, 4, 5, \dots\}$. Then the array would begin with explanation

$$\begin{array}{rlllllll}
Z_1 = \{ & \mathbf{1}, & 2, & 3, & 4, & 5, & 6, & \dots\} \\
Z_2 = \{ & & \mathbf{2}, & & 4, & & 6, & \dots\} \\
Z_3 = \{ & 1, & 2, & & & 5 & & \} \\
Z_4 = \{ & & & 3, & \mathbf{4}, & 5, & 6, & \dots\} \\
 & \vdots & & & & \ddots & &
\end{array}$$

Then $\overline{Z}$ is the set obtained by going down the diagonal, leaving out any numbers that appear along the diagonal and include those j where the array has a gap in the j-th row/column. In the above case, we would leave out 1 and 2, include 3, leave out 4, etc.

Problem siz.1. Show that $\wp(\mathbb{N})$ is non-enumerable by a diagonal argument.

Problem siz.2. Show that the set of functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ is non-enumerable by an explicit diagonal argument. That is, show that if f_1, f_2, …, is a list of functions and each $f_i : \mathbb{Z}^+ \to \mathbb{Z}^+$, then there is some $\overline{f} : \mathbb{Z}^+ \to \mathbb{Z}^+$ not on this list.

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Bibliography