set.1 Equinumerous Sets

sfr:set:equ:sec We have an intuitive notion of "size" of sets, which works fine for finite sets. But what about infinite sets? If we want to come up with a formal way of comparing the sizes of two sets of *any* size, it is a good idea to start with defining when sets are the same size. Let's say sets of the same size are *equinumerous.* We want the formal notion of equinumerosity to correspond with our intuitive notion of "same size," hence the formal notion ought to satisfy the following properties: intro

Reflexivity: Every set is equinumerous with itself.

Symmetry: For any sets X and Y, if X is equinumerous with Y, then Y is equinumerous with X.

Transitivity: For any sets X, Y, and Z, if X is equinumerous with Y and Y is equinumerous with Z, then X is equinumerous with Z.

In other words, we want equinumerosity to be an *equivalence relation.*

Definition set.1. A set X is *equinumerous* with a set Y, $X \approx Y$, if and only if there is a bijective $f \colon X \to Y$.

Proposition set.2. *Equinumerosity defines an equivalence relation.*

Proof. Let X, Y, and Z be sets.

Reflexivity: Using the identity map $1_X \colon X \to X$, where $1_X(x) = x$ for all $x \in X$, we see that X is equinumerous with itself (clearly, 1_X is bijective).

Symmetry: Suppose that X is equinumerous with Y. Then there is a bijective $f \colon X \to Y$. Since f is bijective, its inverse f^{-1} exists and also bijective. Hence, $f^{-1} \colon Y \to X$ is a bijective function from Y to X, so Y is also equinumerous with X.

Transitivity: Suppose that X is equinumerous with Y via the bijective function $f \colon X \to Y$ and that Y is equinumerous with Z via the bijective function $g \colon Y \to Z$. Then the composition of $g \circ f \colon X \to Z$ is bijective, and X is thus equinumerous with Z.

Therefore, equinumerosity is an equivalence relation. □

Theorem set.3. *Suppose X and Y are equinumerous. Then X is enumerable if and only if Y is.*

Proof. Let X and Y be equinumerous. Suppose that X is enumerable. Then either $X = \emptyset$ or there is a surjective function $f \colon \mathbb{Z}^+ \to X$. Since X and Y are equinumerous, there is a bijective $g \colon X \to Y$. If $X = \emptyset$, then $Y = \emptyset$ also (otherwise there would be an element $y \in Y$ but no $x \in X$ with $g(x) = y$). If, on the other hand, $f \colon \mathbb{Z}^+ \to X$ is surjective, then $g \circ f \colon \mathbb{Z}^+ \to Y$ is surjective.

To see this, let $y \in Y$. Since g is surjective, there is an $x \in X$ such that $g(x) = y$. Since f is surjective, there is an $n \in \mathbb{Z}^+$ such that $f(n) = x$. Hence,

$$(g \circ f)(n) = g(f(n)) = g(x) = y$$

and thus $g \circ f$ is surjective. We have that $g \circ f$ is an enumeration of Y, and so Y is enumerable. $\square$

Problem set.1. Show that if X is equinumerous with U and and Y is equinumerous with V, and the intersections $X \cap Y$ and $U \cap V$ are empty, then the unions $X \cup Y$ and $U \cup V$ are equinumerous.

Problem set.2. Show that if X is infinite and enumerable, then it is equinumerous with the positive integers $\mathbb{Z}^+$.

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Bibliography