siz.1 Equinumerosity

sfr:siz:equ:sec

We have an intuitive notion of "size" of sets, which works fine for finite sets. But what about infinite sets? If we want to come up with a formal way of comparing the sizes of two sets of *any* size, it is a good idea to start by defining when sets are the same size. Here is Frege:

> If a waiter wants to be sure that he has laid exactly as many knives as plates on the table, he does not need to count either of them, if he simply lays a knife to the right of each plate, so that every knife on the table lies to the right of some plate. The plates and knives are thus uniquely correlated to each other, and indeed through that same spatial relationship. (Frege, 1884, §70)

The insight of this passage can be brought out through a formal definition:

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Definition siz.1. A is *equinumerous* with B, written $A \approx B$, iff there is a bijection $f\colon A \to B$.

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Proposition siz.2. *Equinumerosity is an equivalence relation.*

Proof. We must show that equinumerosity is reflexive, symmetric, and transitive. Let A, B, and C be sets.

Reflexivity. The identity map $\mathrm{Id}_A\colon A \to A$, where $\mathrm{Id}_A(x) = x$ for all $x \in A$, is a bijection. So $A \approx A$.

Symmetry. Suppose $A \approx B$, i.e., there is a bijection $f\colon A \to B$. Since f is bijective, its inverse f^{-1} exists and is also bijective. Hence, $f^{-1}\colon B \to A$ is a bijection, so $B \approx A$.

Transitivity. Suppose that $A \approx B$ and $B \approx C$, i.e., there are bijections $f\colon A \to B$ and $g\colon B \to C$. Then the composition $g \circ f\colon A \to C$ is bijective, so that $A \approx C$. □

Proposition siz.3. *If $A \approx B$, then A is enumerable if and only if B is.*

The following proof uses **??** if **??** is included and **??** otherwise.

Proof. Suppose $A \approx B$, so there is some bijection $f\colon A \to B$, and suppose that A is enumerable. Then either $A = \emptyset$ or there is a bijection g whose range is A and whose domain is either $\mathbb{N}$ or an initial sequence of natural numbers. If $A = \emptyset$, then $B = \emptyset$ also (otherwise there would be some $y \in B$ with no $x \in A$ such that $g(x) = y$). So suppose we have our bijection g. Then $f \circ g$ is a bijection with range B and domain the same as that of g (i.e., either $\mathbb{N}$ or an initial segment of it), so that B is enumerable.

If B is enumerable, we obtain that A is enumerable by repeating the argument with the bijection $f^{-1}\colon B \to A$ instead of f. □

Problem siz.1. Show that if $A \approx C$ and $B \approx D$, and $A \cap B = C \cap D = \emptyset$, then $A \cup B \approx C \cup D$.

Problem siz.2. Show that if A is infinite and enumerable, then $A \approx \mathbb{N}$.

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Bibliography

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