

## set.1 Equinumerous Sets

sfr:set:equ:  
sec

We have an intuitive notion of “size” of sets, which works fine for finite sets. But what about infinite sets? If we want to come up with a formal way of comparing the sizes of two sets of *any* size, it is a good idea to start with defining when sets are the same size. Let’s say sets of the same size are *equinumerous*. We want the formal notion of equinumerosity to correspond with our intuitive notion of “same size,” hence the formal notion ought to satisfy the following properties:

**Reflexivity:** Every set is equinumerous with itself.

**Symmetry:** For any sets  $X$  and  $Y$ , if  $X$  is equinumerous with  $Y$ , then  $Y$  is equinumerous with  $X$ .

**Transitivity:** For any sets  $X, Y$ , and  $Z$ , if  $X$  is equinumerous with  $Y$  and  $Y$  is equinumerous with  $Z$ , then  $X$  is equinumerous with  $Z$ .

In other words, we want equinumerosity to be an *equivalence relation*.

**Definition set.1.** A set  $X$  is *equinumerous* with a set  $Y$ ,  $X \approx Y$ , if and only if there is a **bijective**  $f: X \rightarrow Y$ .

**Proposition set.2.** *Equinumerosity defines an equivalence relation.*

*Proof.* Let  $X, Y$ , and  $Z$  be sets.

**Reflexivity:** Using the identity map  $1_X: X \rightarrow X$ , where  $1_X(x) = x$  for all  $x \in X$ , we see that  $X$  is equinumerous with itself (clearly,  $1_X$  is **bijective**).

**Symmetry:** Suppose that  $X$  is equinumerous with  $Y$ . Then there is a **bijective**  $f: X \rightarrow Y$ . Since  $f$  is **bijective**, its inverse  $f^{-1}$  exists and also **bijective**. Hence,  $f^{-1}: Y \rightarrow X$  is a **bijective** function from  $Y$  to  $X$ , so  $Y$  is also equinumerous with  $X$ .

**Transitivity:** Suppose that  $X$  is equinumerous with  $Y$  via the **bijective** function  $f: X \rightarrow Y$  and that  $Y$  is equinumerous with  $Z$  via the **bijective** function  $g: Y \rightarrow Z$ . Then the composition of  $g \circ f: X \rightarrow Z$  is **bijective**, and  $X$  is thus equinumerous with  $Z$ .

Therefore, equinumerosity is an equivalence relation. □

**Theorem set.3.** *Suppose  $X$  and  $Y$  are equinumerous. Then  $X$  is **enumerable** if and only if  $Y$  is.*

*Proof.* Let  $X$  and  $Y$  be equinumerous. Suppose that  $X$  is **enumerable**. Then either  $X = \emptyset$  or there is a **surjective** function  $f: \mathbb{Z}^+ \rightarrow X$ . Since  $X$  and  $Y$  are equinumerous, there is a **bijective**  $g: X \rightarrow Y$ . If  $X = \emptyset$ , then  $Y = \emptyset$  also (otherwise there would be an **element**  $y \in Y$  but no  $x \in X$  with  $g(x) = y$ ). If, on the other hand,  $f: \mathbb{Z}^+ \rightarrow X$  is **surjective**, then  $g \circ f: \mathbb{Z}^+ \rightarrow Y$  is **surjective**.

To see this, let  $y \in Y$ . Since  $g$  is **surjective**, there is an  $x \in X$  such that  $g(x) = y$ . Since  $f$  is **surjective**, there is an  $n \in \mathbb{Z}^+$  such that  $f(n) = x$ . Hence,

$$(g \circ f)(n) = g(f(n)) = g(x) = y$$

and thus  $g \circ f$  is **surjective**. We have that  $g \circ f$  is an enumeration of  $Y$ , and so  $Y$  is **enumerable**.  $\square$

**Problem set.1.** Show that if  $X$  is equinumerous with  $U$  and  $Y$  is equinumerous with  $V$ , and the intersections  $X \cap Y$  and  $U \cap V$  are empty, then the unions  $X \cup Y$  and  $U \cup V$  are equinumerous.

**Problem set.2.** Show that if  $X$  is infinite and **enumerable**, then it is equinumerous with the positive integers  $\mathbb{Z}^+$ .

## Photo Credits

## Bibliography