

set.1 Equinumerous Sets

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We have an intuitive notion of “size” of sets, which works fine for finite sets. But what about infinite sets? If we want to come up with a formal way of comparing the sizes of two sets of *any* size, it is a good idea to start with defining when sets are the same size. Let’s say sets of the same size are *equinumerous*. We want the formal notion of equinumerosity to correspond with our intuitive notion of “same size,” hence the formal notion ought to satisfy the following properties:

Reflexivity: Every set is equinumerous with itself.

Symmetry: For any sets X and Y , if X is equinumerous with Y , then Y is equinumerous with X .

Transitivity: For any sets X, Y , and Z , if X is equinumerous with Y and Y is equinumerous with Z , then X is equinumerous with Z .

In other words, we want equinumerosity to be an *equivalence relation*.

Definition set.1. A set X is *equinumerous* with a set Y , $X \approx Y$, if and only if there is a **bijective** $f: X \rightarrow Y$.

Proposition set.2. *Equinumerosity defines an equivalence relation.*

Proof. Let X, Y , and Z be sets.

Reflexivity: Using the identity map $1_X: X \rightarrow X$, where $1_X(x) = x$ for all $x \in X$, we see that X is equinumerous with itself (clearly, 1_X is **bijective**).

Symmetry: Suppose that X is equinumerous with Y . Then there is a **bijective** $f: X \rightarrow Y$. Since f is **bijective**, its inverse f^{-1} exists and also **bijective**. Hence, $f^{-1}: Y \rightarrow X$ is a **bijective** function from Y to X , so Y is also equinumerous with X .

Transitivity: Suppose that X is equinumerous with Y via the **bijective** function $f: X \rightarrow Y$ and that Y is equinumerous with Z via the **bijective** function $g: Y \rightarrow Z$. Then the composition of $g \circ f: X \rightarrow Z$ is **bijective**, and X is thus equinumerous with Z .

Therefore, equinumerosity is an equivalence relation. □

Theorem set.3. *Suppose X and Y are equinumerous. Then X is **enumerable** if and only if Y is.*

Proof. Let X and Y be equinumerous. Suppose that X is **enumerable**. Then either $X = \emptyset$ or there is a **surjective** function $f: \mathbb{Z}^+ \rightarrow X$. Since X and Y are equinumerous, there is a **bijective** $g: X \rightarrow Y$. If $X = \emptyset$, then $Y = \emptyset$ also (otherwise there would be an **element** $y \in Y$ but no $x \in X$ with $g(x) = y$). If, on the other hand, $f: \mathbb{Z}^+ \rightarrow X$ is **surjective**, then $g \circ f: \mathbb{Z}^+ \rightarrow Y$ is **surjective**.

To see this, let $y \in Y$. Since g is **surjective**, there is an $x \in X$ such that $g(x) = y$. Since f is **surjective**, there is an $n \in \mathbb{Z}^+$ such that $f(n) = x$. Hence,

$$(g \circ f)(n) = g(f(n)) = g(x) = y$$

and thus $g \circ f$ is **surjective**. We have that $g \circ f$ is an enumeration of Y , and so Y is **enumerable**. \square

Problem set.1. Show that if X is equinumerous with U and Y is equinumerous with V , and the intersections $X \cap Y$ and $U \cap V$ are empty, then the unions $X \cup Y$ and $U \cup V$ are equinumerous.

Problem set.2. Show that if X is infinite and **enumerable**, then it is equinumerous with the positive integers \mathbb{Z}^+ .

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Bibliography