

siz.1 Enumerable Sets

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One way of specifying a finite set is by listing its **elements**. But conversely, since there are only finitely many **elements** in a set, every finite set can be enumerated. By this we mean: its elements can be put into a list (a list with a beginning, where each **element** of the list other than the first has a unique predecessor). Some infinite sets can also be enumerated, such as the set of positive integers.

Definition siz.1 (Enumeration). Informally, an *enumeration* of a set X is a list (possibly infinite) of **elements** of X such that every **element** of X appears on the list at some finite position. If X has an enumeration, then X is said to be *enumerable*. If X is **enumerable** and infinite, we say X is **denumerable**.

A couple of points about enumerations:

explanation

1. We count as enumerations only lists which have a beginning and in which every **element** other than the first has a single **element** immediately preceding it. In other words, there are only finitely many elements between the first **element** of the list and any other **element**. In particular, this means that every **element** of an enumeration has a finite position: the first **element** has position 1, the second position 2, etc.
2. We can have different enumerations of the same set X which differ by the order in which the **elements** appear: 4, 1, 25, 16, 9 enumerates the (set of the) first five square numbers just as well as 1, 4, 9, 16, 25 does.
3. Redundant enumerations are still enumerations: 1, 1, 2, 2, 3, 3, ... enumerates the same set as 1, 2, 3, ... does.
4. Order and redundancy *do* matter when we specify an enumeration: we can enumerate the positive integers beginning with 1, 2, 3, 1, ..., but the pattern is easier to see when enumerated in the standard way as 1, 2, 3, 4, ...
5. Enumerations must have a beginning: ..., 3, 2, 1 is not an enumeration of the positive integers because it has no first **element**. To see how this follows from the informal definition, ask yourself, "at what position in the list does the number 76 appear?"
6. The following is not an enumeration of the positive integers: 1, 3, 5, ..., 2, 4, 6, ... The problem is that the even numbers occur at places $\infty + 1$, $\infty + 2$, $\infty + 3$, rather than at finite positions.
7. Lists may be gappy: 2, -, 4, -, 6, -, ... enumerates the even positive integers.
8. The empty set is enumerable: it is enumerated by the empty list!

Proposition siz.2. *If X has an enumeration, it has an enumeration without gaps or repetitions.*

Proof. Suppose X has an enumeration x_1, x_2, \dots in which each x_i is an **element** of X or a gap. We can remove repetitions from an enumeration by replacing repeated **elements** by gaps. For instance, we can turn the enumeration into a new one in which x'_i is x_i if x_i is an **element** of X that is not among x_1, \dots, x_{i-1} or is – if it is. We can remove gaps by closing up the elements in the list. To make precise what “closing up” amounts to is a bit difficult to describe. Roughly, it means that we can generate a new enumeration x''_1, x''_2, \dots , where each x''_i is the first **element** in the enumeration x'_1, x'_2, \dots after x'_{i-1} (if there is one). □

The last argument shows that in order to get a good handle on enumerations and **enumerable** sets and to prove things about them, we need a more precise definition. The following provides it.

Definition siz.3 (Enumeration). An *enumeration* of a set X is any **surjective** function $f: \mathbb{Z}^+ \rightarrow X$.

explanation

Let’s convince ourselves that the formal definition and the informal definition using a possibly gappy, possibly infinite list are equivalent. A **surjective** function (partial or total) from \mathbb{Z}^+ to a set X enumerates X . Such a function determines an enumeration as defined informally above: the list $f(1), f(2), f(3), \dots$. Since f is **surjective**, every **element** of X is guaranteed to be the value of $f(n)$ for some $n \in \mathbb{Z}^+$. Hence, every **element** of X appears at some finite position in the list. Since the function may not be **injective**, the list may be redundant, but that is acceptable (as noted above).

On the other hand, given a list that enumerates all **elements** of X , we can define a **surjective** function $f: \mathbb{Z}^+ \rightarrow X$ by letting $f(n)$ be the n th **element** of the list that is not a gap, or the final **element** of the list if there is no n th **element**. There is one case in which this does not produce a **surjective** function: if X is empty, and hence the list is empty. So, every non-empty list determines a **surjective** function $f: \mathbb{Z}^+ \rightarrow X$.

Definition siz.4. A set X is **enumerable** iff it is empty or has an enumeration.

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Example siz.5. A function enumerating the positive integers (\mathbb{Z}^+) is simply the identity function given by $f(n) = n$. A function enumerating the natural numbers \mathbb{N} is the function $g(n) = n - 1$.

Problem siz.1. According to **Definition siz.4**, a set X is enumerable iff $X = \emptyset$ or there is a **surjective** $f: \mathbb{Z}^+ \rightarrow X$. It is also possible to define “**enumerable set**” precisely by: a set is enumerable iff there is an **injective** function $g: X \rightarrow \mathbb{Z}^+$. Show that the definitions are equivalent, i.e., show that there is an **injective** function $g: X \rightarrow \mathbb{Z}^+$ iff either $X = \emptyset$ or there is a **surjective** $f: \mathbb{Z}^+ \rightarrow X$.

Example siz.6. The functions $f: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ and $g: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ given by

$$f(n) = 2n \text{ and}$$

$$g(n) = 2n + 1$$

enumerate the even positive integers and the odd positive integers, respectively. However, neither function is an enumeration of \mathbb{Z}^+ , since neither is **surjective**.

Problem siz.2. Define an enumeration of the positive squares 4, 9, 16, ...

Example siz.7. The function $f(n) = (-1)^n \lceil \frac{n-1}{2} \rceil$ (where $\lceil x \rceil$ denotes the *ceiling* function, which rounds x up to the nearest integer) enumerates the set of integers \mathbb{Z} . Notice how f generates the values of \mathbb{Z} by “hopping” back and forth between positive and negative integers:

$$\begin{array}{cccccccc} f(1) & f(2) & f(3) & f(4) & f(5) & f(6) & f(7) & \dots \\ -\lceil \frac{0}{2} \rceil & \lceil \frac{1}{2} \rceil & -\lceil \frac{2}{2} \rceil & \lceil \frac{3}{2} \rceil & -\lceil \frac{4}{2} \rceil & \lceil \frac{5}{2} \rceil & -\lceil \frac{6}{2} \rceil & \dots \\ 0 & 1 & -1 & 2 & -2 & 3 & \dots & \end{array}$$

You can also think of f as defined by cases as follows:

$$f(n) = \begin{cases} 0 & \text{if } n = 1 \\ n/2 & \text{if } n \text{ is even} \\ -(n-1)/2 & \text{if } n \text{ is odd and } > 1 \end{cases}$$

Problem siz.3. Show that if X and Y are **enumerable**, so is $X \cup Y$.

Problem siz.4. Show by induction on n that if X_1, X_2, \dots, X_n are all **enumerable**, so is $X_1 \cup \dots \cup X_n$.

That is fine for “easy” sets. What about the set of, say, pairs of positive integers? [explanation](#)

$$\mathbb{Z}^+ \times \mathbb{Z}^+ = \{\langle n, m \rangle : n, m \in \mathbb{Z}^+\}$$

We can organize the pairs of positive integers in an *array*, such as the following:

	1	2	3	4	...
1	$\langle 1, 1 \rangle$	$\langle 1, 2 \rangle$	$\langle 1, 3 \rangle$	$\langle 1, 4 \rangle$...
2	$\langle 2, 1 \rangle$	$\langle 2, 2 \rangle$	$\langle 2, 3 \rangle$	$\langle 2, 4 \rangle$...
3	$\langle 3, 1 \rangle$	$\langle 3, 2 \rangle$	$\langle 3, 3 \rangle$	$\langle 3, 4 \rangle$...
4	$\langle 4, 1 \rangle$	$\langle 4, 2 \rangle$	$\langle 4, 3 \rangle$	$\langle 4, 4 \rangle$...
\vdots	\vdots	\vdots	\vdots	\vdots	\ddots

Clearly, every ordered pair in $\mathbb{Z}^+ \times \mathbb{Z}^+$ will appear exactly once in the array. In particular, $\langle n, m \rangle$ will appear in the n th column and m th row. But how do

we organize the elements of such an array into a one-way list? The pattern in the array below demonstrates one way to do this:

	1	2	4	7	...
	3	5	8
	6	9
	10
	⋮	⋮	⋮	⋮	⋮

This pattern is called *Cantor's zig-zag method*. Other patterns are perfectly permissible, as long as they “zig-zag” through every cell of the array. By Cantor's zig-zag method, the enumeration for $\mathbb{Z}^+ \times \mathbb{Z}^+$ according to this scheme would be:

$$\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 1, 3 \rangle, \langle 2, 2 \rangle, \langle 3, 1 \rangle, \langle 1, 4 \rangle, \langle 2, 3 \rangle, \langle 3, 2 \rangle, \langle 4, 1 \rangle, \dots$$

What ought we do about enumerating, say, the set of ordered triples of positive integers?

$$\mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+ = \{ \langle n, m, k \rangle : n, m, k \in \mathbb{Z}^+ \}$$

We can think of $\mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+$ as the Cartesian product of $\mathbb{Z}^+ \times \mathbb{Z}^+$ and \mathbb{Z}^+ , that is,

$$(\mathbb{Z}^+)^3 = (\mathbb{Z}^+ \times \mathbb{Z}^+) \times \mathbb{Z}^+ = \{ \langle \langle n, m \rangle, k \rangle : \langle n, m \rangle \in \mathbb{Z}^+ \times \mathbb{Z}^+, k \in \mathbb{Z}^+ \}$$

and thus we can enumerate $(\mathbb{Z}^+)^3$ with an array by labelling one axis with the enumeration of \mathbb{Z}^+ , and the other axis with the enumeration of $(\mathbb{Z}^+)^2$:

	1	2	3	4	...
⟨1, 1⟩	⟨1, 1, 1⟩	⟨1, 1, 2⟩	⟨1, 1, 3⟩	⟨1, 1, 4⟩	...
⟨1, 2⟩	⟨1, 2, 1⟩	⟨1, 2, 2⟩	⟨1, 2, 3⟩	⟨1, 2, 4⟩	...
⟨2, 1⟩	⟨2, 1, 1⟩	⟨2, 1, 2⟩	⟨2, 1, 3⟩	⟨2, 1, 4⟩	...
⟨1, 3⟩	⟨1, 3, 1⟩	⟨1, 3, 2⟩	⟨1, 3, 3⟩	⟨1, 3, 4⟩	...
⋮	⋮	⋮	⋮	⋮	⋮

Thus, by using a method like Cantor's zig-zag method, we may similarly obtain an enumeration of $(\mathbb{Z}^+)^3$.

Cantor's zig-zag method makes the enumerability of $(\mathbb{Z}^+)^2$ (and analogously, $(\mathbb{Z}^+)^3$, etc.) visually evident. Following the zig-zag line in the array and counting the places, we can tell that $\langle 2, 3 \rangle$ is at place 8, but specifying the inverse $g: (\mathbb{Z}^+)^2 \rightarrow \mathbb{Z}^+$ of the zig-zag enumeration such that

$$g(\langle 1, 1 \rangle) = 1, \quad g(\langle 1, 2 \rangle) = 2, \quad g(\langle 2, 1 \rangle) = 3, \quad \dots \quad g(\langle 2, 3 \rangle) = 8, \quad \dots$$

would be helpful. To calculate the position of each pair in the enumeration, we can use the function below. (The exact derivation of the function is somewhat messy, so we are skipping it here.)

$$g(n, m) = \frac{(n + m - 2)(n + m - 1)}{2} + n$$

Accordingly, the pair $\langle 2, 3 \rangle$ is in position $((2+3-2)(2+3-1)/2)+2 = (3 \cdot 4/2)+2 = (12/2)+2 = 8$; pair $\langle 3, 7 \rangle$ is in position $((3+7-2)(3+7-1)/2)+3 = 39$.

Functions like g above, i.e., inverses of enumerations of $(\mathbb{Z}^+)^2$, are called *pairing functions*.

Definition siz.8 (Pairing function). A function $f: (\mathbb{Z}^+)^2 \rightarrow \mathbb{Z}^+$ is an arithmetical *pairing function* if f is total and injective. We also say that f *encodes* $(\mathbb{Z}^+)^2$, and that for $f(\langle n, m \rangle) = k$, k is the *code* for $\langle n, m \rangle$.

The idea is that we can use such functions to encode pairs of positive integers in \mathbb{Z}^+ , or, in other words, represent pairs of positive integers as positive integers. Using the inverse of the pairing function, we can *decode* the integer, i.e., find out which pair of positive integers is represented. explanation

There are other enumerations of $(\mathbb{Z}^+)^2$ that make it easier to figure out what their inverses are. Here is one. Instead of visualizing the enumeration in an array, start with the list of positive integers associated with (initially) empty spaces. Imagine filling these spaces successively with pairs $\langle n, m \rangle$ as follow. Starting with the pairs that have 1 in the first place (i.e., pairs $\langle 1, m \rangle$), put the first (i.e., $\langle 1, 1 \rangle$) in the first empty place, then skip an empty space, put the second (i.e., $\langle 1, 2 \rangle$) in the next empty place, skip one again, and so forth. The (incomplete) beginning of our enumeration now looks like this

$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	$f(6)$	$f(7)$	$f(8)$	$f(9)$	$f(10)$	\dots
$\langle 1, 1 \rangle$	$\langle 1, 2 \rangle$	$\langle 1, 3 \rangle$	$\langle 1, 4 \rangle$	$\langle 1, 5 \rangle$	\dots					

Repeat this with pairs $\langle 2, m \rangle$ for the place that still remain empty, again skipping every other empty place:

$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	$f(6)$	$f(7)$	$f(8)$	$f(9)$	$f(10)$	\dots
$\langle 1, 1 \rangle$	$\langle 2, 1 \rangle$	$\langle 1, 2 \rangle$	$\langle 1, 3 \rangle$	$\langle 2, 2 \rangle$	$\langle 1, 4 \rangle$	$\langle 1, 5 \rangle$	$\langle 2, 3 \rangle$	\dots		

Enter pairs $\langle 3, m \rangle$, $\langle 4, m \rangle$, etc., in the same way. Our completed enumeration thus starts like this:

$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	$f(6)$	$f(7)$	$f(8)$	$f(9)$	$f(10)$	\dots
$\langle 1, 1 \rangle$	$\langle 2, 1 \rangle$	$\langle 1, 2 \rangle$	$\langle 3, 1 \rangle$	$\langle 1, 3 \rangle$	$\langle 2, 2 \rangle$	$\langle 1, 4 \rangle$	$\langle 4, 1 \rangle$	$\langle 1, 5 \rangle$	$\langle 2, 3 \rangle$	\dots

If we number the cells in the array above according to this enumeration, we will not find a neat zig-zag line, but this arrangement:

	1	2	3	4	5	6	...
1	1	3	5	7	9	11	...
2	2	6	10	14	18
3	4	12	20	28
4	8	24	40
5	16	48
6	32
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

We can see that the pairs in the first row are in the odd numbered places of our enumeration, i.e., pair $\langle 1, m \rangle$ is in place $2m - 1$; pairs in the second row, $\langle 1, m \rangle$, are in places whose number is the double of an odd number, specifically, $2 \cdot (2m - 1)$; pairs in the third row, $\langle 1, m \rangle$, are in places whose number is four times an odd number, $4 \cdot (2m - 1)$; and so on. The factors of $(2m - 1)$ for each row, $1, 2, 4, 8, \dots$, are powers of 2: $2^0, 2^1, 2^2, 2^3, \dots$. In fact, the relevant exponent is one less than the first member of the pair in question. Thus, for pair $\langle n, m \rangle$ the factor is $n - 1$. This gives us the general formula: $2^{n-1} \cdot (2m - 1)$, and hence:

Example siz.9. The function $f: (\mathbb{Z}^+)^2 \rightarrow \mathbb{Z}^+$ given by

$$h(n, m) = 2^{n-1}(2m - 1)$$

is a pairing function for the set of pairs of positive integers $(\mathbb{Z}^+)^2$.

explanation Accordingly, in our second enumeration of $(\mathbb{Z}^+)^2$, the pair $\langle 2, 3 \rangle$ is in position $2^{2-1} \cdot (2 \cdot 3 - 1) = 2 \cdot 5 = 10$; pair $\langle 3, 7 \rangle$ is in position $2^{3-1} \cdot (2 \cdot 7 - 1) = 52$.

Another common pairing function that encodes $(\mathbb{Z}^+)^2$ is the following:

Example siz.10. The function $f: (\mathbb{Z}^+)^2 \rightarrow \mathbb{Z}^+$ given by

$$j(n, m) = 2^n 3^m$$

is a pairing function for the set of pairs of positive integers $(\mathbb{Z}^+)^2$.

explanation j is injective, but not surjective. That means the inverse of j is a partial, surjective function, and hence an enumeration of $(\mathbb{Z}^+)^2$.

Problem siz.5. Give an enumeration of the set of all positive rational numbers. (A positive rational number is one that can be written as a fraction n/m with $n, m \in \mathbb{Z}^+$).

Problem siz.6. Show that \mathbb{Q} is **enumerable**. (A rational number is one that can be written as a fraction z/m with $z \in \mathbb{Z}, m \in \mathbb{Z}^+$).

Problem siz.7. Define an enumeration of \mathbb{B}^* .

Problem siz.8. Recall from your introductory logic course that each possible truth table expresses a truth function. In other words, the truth functions are all functions from $\mathbb{B}^k \rightarrow \mathbb{B}$ for some k . Prove that the set of all truth functions is enumerable.

Problem siz.9. Show that the set of all finite subsets of an arbitrary infinite enumerable set is enumerable.

Problem siz.10. A set of positive integers is said to be *cofinite* iff it is the complement of a finite set of positive integers. Let I be the set that contains all the finite and cofinite sets of positive integers. Show that I is enumerable.

Problem siz.11. Show that the enumerable union of enumerable sets is enumerable. That is, whenever X_1, X_2, \dots are sets, and each X_i is enumerable, then the union $\bigcup_{i=1}^{\infty} X_i$ of all of them is also enumerable.

Problem siz.12. Specify a function that encodes \mathbb{N}^3 .

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Bibliography