

## siz.1 Comparing Sizes of Sets

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Just like we were able to make precise when two sets have the same size in a way that also accounts for the size of infinite sets, we can also compare the sizes of sets in a precise way. Our definition of “is smaller than (or equinumerous)” will require, instead of a **bijection** between the sets, a total **injective** function from the first set to the second. If such a function exists, the size of the first set is less than or equal to the size of the second. Intuitively, an **injective** function from one set to another guarantees that the range of the function has at least as many elements as the domain, since no two **elements** of the domain map to the same **element** of the range.

explanation

**Definition siz.1.**  $X$  is *no larger than*  $Y$ ,  $X \preceq Y$ , if and only if there is an **injective** function  $f: X \rightarrow Y$ .

**Theorem siz.2** (Schröder-Bernstein). *Let  $X$  and  $Y$  be sets. If  $X \preceq Y$  and  $Y \preceq X$ , then  $X \approx Y$ .*

In other words, if there is a total **injective** function from  $X$  to  $Y$ , and if there is a total **injective** function from  $Y$  back to  $X$ , then there is a total **bijection** from  $X$  to  $Y$ . Sometimes, it can be difficult to think of a **bijection** between two equinumerous sets, so the Schröder-Bernstein theorem allows us to break the comparison down into cases so we only have to think of an **injection** from the first to the second, and vice-versa. The Schröder-Bernstein theorem, apart from being convenient, justifies the act of discussing the “sizes” of sets, for it tells us that set cardinalities have the familiar anti-symmetric property that numbers have.

explanation

**Definition siz.3.**  $X$  is *smaller than*  $Y$ ,  $X \prec Y$ , if and only if there is an **injective** function  $f: X \rightarrow Y$  but no **bijection**  $g: X \rightarrow Y$ .

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**Theorem siz.4** (Cantor). *For all  $X$ ,  $X \prec \wp(X)$ .*

*Proof.* The function  $f: X \rightarrow \wp(X)$  that maps any  $x \in X$  to its singleton  $\{x\}$  is **injective**, since if  $x \neq y$  then also  $f(x) = \{x\} \neq \{y\} = f(y)$ .

There cannot be a **surjective** function  $g: X \rightarrow \wp(X)$ , let alone a **bijection** one. For suppose that  $g: X \rightarrow \wp(X)$ . Since  $g$  is total, every  $x \in X$  is mapped to a subset  $g(x) \subseteq X$ . We show that  $g$  cannot be surjective. To do this, we define a subset  $Y \subseteq X$  which by definition cannot be in the range of  $g$ . Let

$$\bar{Y} = \{x \in X : x \notin g(x)\}.$$

Since  $g(x)$  is defined for all  $x \in X$ ,  $\bar{Y}$  is clearly a well-defined subset of  $X$ . But, it cannot be in the range of  $g$ . Let  $x \in X$  be arbitrary, we show that  $\bar{Y} \neq g(x)$ . If  $x \in g(x)$ , then it does not satisfy  $x \notin g(x)$ , and so by the definition of  $\bar{Y}$ , we have  $x \notin \bar{Y}$ . If  $x \in \bar{Y}$ , it must satisfy the defining property of  $\bar{Y}$ , i.e.,  $x \notin g(x)$ . Since  $x$  was arbitrary this shows that for each  $x \in X$ ,  $x \in g(x)$  iff  $x \notin \bar{Y}$ , and so  $g(x) \neq \bar{Y}$ . So  $\bar{Y}$  cannot be in the range of  $g$ , contradicting the assumption that  $g$  is surjective.  $\square$

[explanation](#) It's instructive to compare the proof of [Theorem siz.4](#) to that of [??](#). There we showed that for any list  $Z_1, Z_2, \dots$ , of subsets of  $\mathbb{Z}^+$  one can construct a set  $\bar{Z}$  of numbers guaranteed not to be on the list. It was guaranteed not to be on the list because, for every  $n \in \mathbb{Z}^+$ ,  $n \in Z_n$  iff  $n \notin \bar{Z}$ . This way, there is always some number that is [an element](#) of one of  $Z_n$  and  $\bar{Z}$  but not the other. We follow the same idea here, except the indices  $n$  are now [elements](#) of  $X$  instead of  $\mathbb{Z}^+$ . The set  $\bar{Y}$  is defined so that it is different from  $g(x)$  for each  $x \in X$ , because  $x \in g(x)$  iff  $x \notin \bar{Y}$ . Again, there is always [an element](#) of  $X$  which is [an element](#) of one of  $g(x)$  and  $\bar{Y}$  but not the other. And just as  $\bar{Z}$  therefore cannot be on the list  $Z_1, Z_2, \dots$ ,  $\bar{Y}$  cannot be in the range of  $g$ .

**Problem siz.1.** Show that there cannot be [an injective](#) function  $g: \wp(X) \rightarrow X$ , for any set  $X$ . Hint: Suppose  $g: \wp(X) \rightarrow X$  is [injective](#). Then for each  $x \in X$  there is at most one  $Y \subseteq X$  such that  $g(Y) = x$ . Define a set  $\bar{Y}$  such that for every  $x \in X$ ,  $g(\bar{Y}) \neq x$ .

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## Bibliography