siz.1 Comparing Sizes of Sets

sfr:siz:car:sec Just like we were able to make precise when two sets have the same size in a way that also accounts for the size of infinite sets, we can also compare the sizes of sets in a precise way. Our definition of "is smaller than (or equinumerous)" will require, instead of a bijection between the sets, a total injective function from the first set to the second. If such a function exists, the size of the first set is less than or equal to the size of the second. Intuitively, an injective function from one set to another guarantees that the range of the function has at least as many elements as the domain, since no two elements of the domain map to the same element of the range. explanation

Definition siz.1. X is *no larger than* Y, $X \preceq Y$, if and only if there is an injective function $f\colon X \to Y$.

Theorem siz.2 (Schröder-Bernstein)**.** *Let X and Y be sets. If $X \preceq Y$ and $Y \preceq X$, then $X \approx Y$.*

In other words, if there is a total injective function from X to Y, and if there is a total injective function from Y back to X, then there is a total bijection from X to Y. Sometimes, it can be difficult to think of a bijection between two equinumerous sets, so the Schröder-Bernstein theorem allows us to break the comparison down into cases so we only have to think of an injection from the first to the second, and vice-versa. The Schröder-Bernstein theorem, apart from being convenient, justifies the act of discussing the "sizes" of sets, for it tells us that set cardinalities have the familiar anti-symmetric property that numbers have. explanation

Definition siz.3. X is *smaller than* Y, $X \prec Y$, if and only if there is an injective function $f\colon X \to Y$ but no bijective $g\colon X \to Y$.

sfr:siz:car:thm:cantor **Theorem siz.4** (Cantor)**.** *For all X, $X \prec \wp(X)$.*

Proof. The function $f\colon X \to \wp(X)$ that maps any $x \in X$ to its singleton $\{x\}$ is injective, since if $x \neq y$ then also $f(x) = \{x\} \neq \{y\} = f(y)$.

There cannot be a surjective function $g\colon X \to \wp(X)$, let alone a bijective one. For suppose that $g\colon X \to \wp(X)$. Since g is total, every $x \in X$ is mapped to a subset $g(x) \subseteq X$. We show that g cannot be surjective. To do this, we define a subset $Y \subseteq X$ which by definition cannot be in the range of g. Let

$$\overline{Y} = \{x \in X : x \notin g(x)\}.$$

Since $g(x)$ is defined for all $x \in X$, $\overline{Y}$ is clearly a well-defined subset of X. But, it cannot be in the range of g. Let $x \in X$ be arbitrary, we show that $\overline{Y} \neq g(x)$. If $x \in g(x)$, then it does not satisfy $x \notin g(x)$, and so by the definition of $\overline{Y}$, we have $x \notin \overline{Y}$. If $x \in \overline{Y}$, it must satisfy the defining property of $\overline{Y}$, i.e., $x \notin g(x)$. Since x was arbitrary this shows that for each $x \in X$, $x \in g(x)$ iff $x \notin \overline{Y}$, and so $g(x) \neq \overline{Y}$. So $\overline{Y}$ cannot be in the range of g, contradicting the assumption that g is surjective. □

explanation

It's instructive to compare the proof of Theorem siz.4 to that of **??**. There we showed that for any list Z_1, Z_2, ..., of subsets of $\mathbb{Z}^+$ one can construct a set $\overline{Z}$ of numbers guaranteed not to be on the list. It was guaranteed not to be on the list because, for every $n \in \mathbb{Z}^+$, $n \in Z_n$ iff $n \notin \overline{Z}$. This way, there is always some number that is an element of one of Z_n and $\overline{Z}$ but not the other. We follow the same idea here, except the indices n are now elements of X instead of $\mathbb{Z}^+$. The set $\overline{Y}$ is defined so that it is different from $g(x)$ for each $x \in X$, because $x \in g(x)$ iff $x \notin \overline{Y}$. Again, there is always an element of X which is an element of one of $g(x)$ and $\overline{Y}$ but not the other. And just as $\overline{Z}$ therefore cannot be on the list Z_1, Z_2, ..., $\overline{Y}$ cannot be in the range of g.

Problem siz.1. Show that there cannot be an injective function $g\colon \wp(X) \to X$, for any set X. Hint: Suppose $g\colon \wp(X) \to X$ is injective. Then for each $x \in X$ there is at most one $Y \subseteq X$ such that $g(Y) = x$. Define a set $\overline{Y}$ such that for every $x \in X$, $g(\overline{Y}) \neq x$.

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Bibliography