

siz.1 Sets of Different Sizes, and Cantor's Theorem

sfr:siz:car: We have offered a precise statement of the idea that two sets have the same size. explanation
sec

We can also offer a precise statement of the idea that one set is smaller than another. Our definition of “is smaller than (or equinumerous)” will require, instead of a **bijection** between the sets, an **injection** from the first set to the second. If such a function exists, the size of the first set is less than or equal to the size of the second. Intuitively, an **injection** from one set to another guarantees that the range of the function has at least as many **elements** as the domain, since no two **elements** of the domain map to the same **element** of the range.

Definition siz.1. A is *no larger than* B , written $A \preceq B$, iff there is an **injection** $f: A \rightarrow B$.

It is clear that this is a reflexive and transitive relation, but that it is not symmetric (this is left as an exercise). We can also introduce a notion, which states that one set is (strictly) smaller than another.

Definition siz.2. A is *smaller than* B , written $A \prec B$, iff there is an **injection** $f: A \rightarrow B$ but no **bijection** $g: A \rightarrow B$, i.e., $A \preceq B$ and $A \not\approx B$.

It is clear that this relation is irreflexive and transitive. (This is left as an exercise.) Using this notation, we can say that a set A is **enumerable** iff $A \preceq \mathbb{N}$, and that A is **non-enumerable** iff $\mathbb{N} \prec A$. This allows us to restate ?? as the observation that $\mathbb{Z}^+ \prec \wp(\mathbb{Z}^+)$. In fact, **Cantor (1892)** proved that this last point is *perfectly general*:

sfr:siz:car: **Theorem siz.3 (Cantor).** $A \prec \wp(A)$, for any set A .
thm:cantor

Proof. The map $f(x) = \{x\}$ is an **injection** $f: A \rightarrow \wp(A)$, since if $x \neq y$, then also $\{x\} \neq \{y\}$ by extensionality, and so $f(x) \neq f(y)$. So we have that $A \preceq \wp(A)$.

We present the slow proof if ?? is present, otherwise a faster proof matching ??.

It remains to show that $A \not\approx \wp(A)$. For reductio, suppose $A \approx \wp(A)$, i.e., there is some **bijection** $g: A \rightarrow \wp(A)$. Now consider:

$$D = \{x \in A : x \notin g(x)\}$$

Note that $D \subseteq A$, so that $D \in \wp(A)$. Since g is a **bijection**, there is some $y \in A$ such that $g(y) = D$. But now we have:

$$y \in g(y) \text{ iff } y \in D \text{ iff } y \notin g(y).$$

This is a contradiction; so $A \not\approx \wp(A)$. □

[explanation](#) The proof is also worth comparing with the proof of Russell's Paradox, ??
Indeed, Cantor's Theorem was the inspiration for Russell's own paradox.

Problem siz.1. Show that there cannot be an injection $g: \wp(A) \rightarrow A$, for any set A . Hint: Suppose $g: \wp(A) \rightarrow A$ is injective. Consider $D = \{g(B) : B \subseteq A \text{ and } g(B) \notin B\}$. Let $x = g(D)$. Use the fact that g is injective to derive a contradiction.

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Bibliography

Cantor, Georg. 1892. Über eine elementare Frage der Mannigfaltigkeitslehre.
Jahresbericht der deutschen Mathematiker-Vereinigung 1: 75–8.