

Figure 1: The union $X \cup Y$ of two sets is set of **elements** of X together with those of Y .

set.1 Unions and Intersections

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We can define new sets by abstraction, and the property used to define the new set can mention sets we've already defined. So for instance, if X and Y are sets, the set $\{x : x \in X \vee x \in Y\}$ defines a set which consists of all those objects which are **elements** of either X or Y , i.e., it's the set that combines the **elements** of X and Y . This operation on sets—combining them—is very useful and common, and so we give it a name and a symbol.

explanation

Definition set.1 (Union). The *union* of two sets X and Y , written $X \cup Y$, is the set of all things which are **elements** of X , Y , or both.

$$X \cup Y = \{x : x \in X \vee x \in Y\}$$

Example set.2. Since the multiplicity of **elements** doesn't matter, the union of two sets which have **an element** in common contains that **element** only once, e.g., $\{a, b, c\} \cup \{a, 0, 1\} = \{a, b, c, 0, 1\}$.

The union of a set and one of its subsets is just the bigger set: $\{a, b, c\} \cup \{a\} = \{a, b, c\}$.

The union of a set with the empty set is identical to the set: $\{a, b, c\} \cup \emptyset = \{a, b, c\}$.

Problem set.1. Prove rigorously that if $X \subseteq Y$, then $X \cup Y = Y$.

The operation that forms the set of all **elements** that X and Y have in common is called their *intersection*.

Definition set.3 (Intersection). The *intersection* of two sets X and Y , written $X \cap Y$, is the set of all things which are **elements** of both X and Y .

$$X \cap Y = \{x : x \in X \wedge x \in Y\}$$

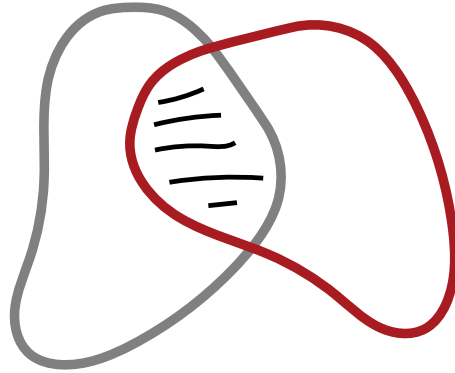


Figure 2: The intersection $X \cap Y$ of two sets is the set of **elements** they have in common.

Two sets are called *disjoint* if their intersection is empty. This means they have no **elements** in common.

Example set.4. If two sets have no **elements** in common, their intersection is empty: $\{a, b, c\} \cap \{0, 1\} = \emptyset$.

If two sets do have **elements** in common, their intersection is the set of all those: $\{a, b, c\} \cap \{a, b, d\} = \{a, b\}$.

The intersection of a set with one of its subsets is just the smaller set: $\{a, b, c\} \cap \{a, b\} = \{a, b\}$.

The intersection of any set with the empty set is empty: $\{a, b, c\} \cap \emptyset = \emptyset$.

Problem set.2. Prove rigorously that if $X \subseteq Y$, then $X \cap Y = X$.

explanation

We can also form the union or intersection of more than two sets. An elegant way of dealing with this in general is the following: suppose you collect all the sets you want to form the union (or intersection) of into a single set. Then we can define the union of all our original sets as the set of all objects which belong to at least one **element** of the set, and the intersection as the set of all objects which belong to every **element** of the set.

Definition set.5. If Z is a set of sets, then $\bigcup Z$ is the set of **elements** of **elements** of Z :

$$\bigcup Z = \{x : x \text{ belongs to an element of } Z\}, \text{ i.e.,}$$

$$\bigcup Z = \{x : \text{there is a } Y \in Z \text{ so that } x \in Y\}$$

Definition set.6. If Z is a set of sets, then $\bigcap Z$ is the set of objects which all elements of Z have in common:

$$\bigcap Z = \{x : x \text{ belongs to every element of } Z\}, \text{ i.e.,}$$

$$\bigcap Z = \{x : \text{for all } Y \in Z, x \in Y\}$$

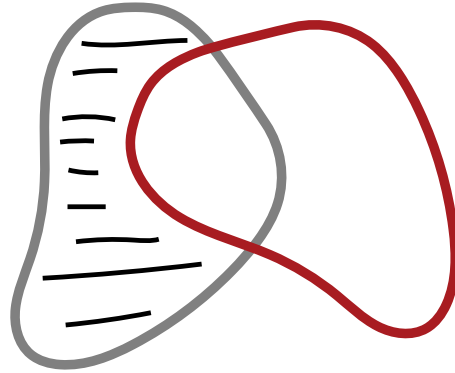


Figure 3: The difference $X \setminus Y$ of two sets is the set of those **elements** of X which are not also elements of Y .

Example set.7. Suppose $Z = \{\{a, b\}, \{a, d, e\}, \{a, d\}\}$. Then $\bigcup Z = \{a, b, d, e\}$ and $\bigcap Z = \{a\}$.

We could also do the same for a sequence of sets X_1, X_2, \dots

$$\bigcup_i X_i = \{x : x \text{ belongs to one of the } X_i\}$$
$$\bigcap_i X_i = \{x : x \text{ belongs to every } X_i\}.$$

Definition set.8 (Difference). The *difference* $X \setminus Y$ is the set of all **elements** of X which are not also **elements** of Y , i.e.,

$$X \setminus Y = \{x : x \in X \text{ and } x \notin Y\}.$$

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Bibliography