

Figure 1: The union $A \cup B$ of two sets is set of **elements** of A together with those of B .

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fig:union

set.1 Unions and Intersections

In ??, we introduced definitions of sets by abstraction, i.e., definitions of the form $\{x : \varphi(x)\}$. Here, we invoke some property φ , and this property can mention sets we've already defined. So for instance, if A and B are sets, the set $\{x : x \in A \vee x \in B\}$ consists of all those objects which are **elements** of either A or B , i.e., it's the set that combines the **elements** of A and B . We can visualize this as in **Figure 1**, where the highlighted area indicates the **elements** of the two sets A and B together.

This operation on sets—combining them—is very useful and common, and so we give it a formal name and a symbol.

Definition set.1 (Union). The *union* of two sets A and B , written $A \cup B$, is the set of all things which are **elements** of A , B , or both.

$$A \cup B = \{x : x \in A \vee x \in B\}$$

Example set.2. Since the multiplicity of **elements** doesn't matter, the union of two sets which have **an element** in common contains that **element** only once, e.g., $\{a, b, c\} \cup \{a, 0, 1\} = \{a, b, c, 0, 1\}$.

The union of a set and one of its subsets is just the bigger set: $\{a, b, c\} \cup \{a\} = \{a, b, c\}$.

The union of a set with the empty set is identical to the set: $\{a, b, c\} \cup \emptyset = \{a, b, c\}$.

Problem set.1. Prove that if $A \subseteq B$, then $A \cup B = B$.

We can also consider a “dual” operation to union. This is the operation that forms the set of all **elements** that are **elements** of A and are also **elements** of B . This operation is called *intersection*, and can be depicted as in **Figure 2**.

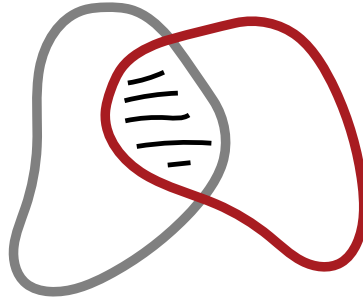


Figure 2: The intersection $A \cap B$ of two sets is the set of **elements** they have in common.

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fig:intersection

Definition set.3 (Intersection). The *intersection* of two sets A and B , written $A \cap B$, is the set of all things which are **elements** of both A and B .

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

Two sets are called *disjoint* if their intersection is empty. This means they have no **elements** in common.

Example set.4. If two sets have no **elements** in common, their intersection is empty: $\{a, b, c\} \cap \{0, 1\} = \emptyset$.

If two sets do have **elements** in common, their intersection is the set of all those: $\{a, b, c\} \cap \{a, b, d\} = \{a, b\}$.

The intersection of a set with one of its subsets is just the smaller set: $\{a, b, c\} \cap \{a, b\} = \{a, b\}$.

The intersection of any set with the empty set is empty: $\{a, b, c\} \cap \emptyset = \emptyset$.

Problem set.2. Prove rigorously that if $A \subseteq B$, then $A \cap B = A$.

explanation

We can also form the union or intersection of more than two sets. An elegant way of dealing with this in general is the following: suppose you collect all the sets you want to form the union (or intersection) of into a single set. Then we can define the union of all our original sets as the set of all objects which belong to at least one **element** of the set, and the intersection as the set of all objects which belong to every **element** of the set.

Definition set.5. If A is a set of sets, then $\bigcup A$ is the set of **elements** of **elements** of A :

$$\begin{aligned} \bigcup A &= \{x : x \text{ belongs to an element of } A\}, \text{ i.e.,} \\ &= \{x : \text{there is a } B \in A \text{ so that } x \in B\} \end{aligned}$$

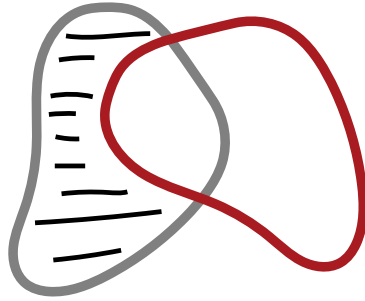


Figure 3: The difference $A \setminus B$ of two sets is the set of those **elements** of A which are not also **elements** of B .

[sfr:set:uni:
difference](#)

Definition set.6. If A is a set of sets, then $\bigcap A$ is the set of objects which all elements of A have in common:

$$\begin{aligned} \bigcap A &= \{x : x \text{ belongs to every element of } A\}, \text{ i.e.,} \\ &= \{x : \text{for all } B \in A, x \in B\} \end{aligned}$$

Example set.7. Suppose $A = \{\{a, b\}, \{a, d, e\}, \{a, d\}\}$. Then $\bigcup A = \{a, b, d, e\}$ and $\bigcap A = \{a\}$.

Problem set.3. Show that if A is a set and $A \in B$, then $A \subseteq \bigcup B$.

We could also do the same for a sequence of sets A_1, A_2, \dots

$$\begin{aligned} \bigcup_i A_i &= \{x : x \text{ belongs to one of the } A_i\} \\ \bigcap_i A_i &= \{x : x \text{ belongs to every } A_i\}. \end{aligned}$$

When we have an *index* of sets, i.e., some set I such that we are considering A_i for each $i \in I$, we may also use these abbreviations:

$$\begin{aligned} \bigcup_{i \in I} A_i &= \bigcup \{A_i : i \in I\} \\ \bigcap_{i \in I} A_i &= \bigcap \{A_i : i \in I\} \end{aligned}$$

Finally, we may want to think about the set of all **elements** in A which are not in B . We can depict this as in **Figure 3**.

Definition set.8 (Difference). The *set difference* $A \setminus B$ is the set of all **elements** of A which are not also **elements** of B , i.e.,

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$

Problem set.4. Prove that if $A \subsetneq B$, then $B \setminus A \neq \emptyset$.

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Bibliography