set.1 Subsets and Power Sets

We will often want to compare sets. And one obvious kind of comparison one might make is as follows: *everything in one set is in the other too*. This situation is sufficiently important for us to introduce some new notation.

**Definition set.1 (Subset).** If every element of a set \( A \) is also an element of \( B \), then we say that \( A \) is a *subset* of \( B \), and write \( A \subseteq B \). If \( A \) is not a subset of \( B \) we write \( A \nsubseteq B \). If \( A \subseteq B \) but \( A \neq B \), we write \( A \subset B \) and say that \( A \) is a *proper subset* of \( B \).

**Example set.2.** Every set is a subset of itself, and \( \emptyset \) is a subset of every set. The set of even numbers is a subset of the set of natural numbers. Also, \( \{a, b\} \subseteq \{a, b, c\} \). But \( \{a, b, e\} \) is not a subset of \( \{a, b, c\} \).

**Example set.3.** The number 2 is an element of the set of integers, whereas the set of even numbers is a subset of the set of integers. However, a set may happen to both be an element and a subset of some other set, e.g., \( \{0\} \in \{0, \{0\}\} \) and also \( \{0\} \subseteq \{0, \{0\}\} \).

Extensionality gives a criterion of identity for sets: \( A = B \) iff every element of \( A \) is also an element of \( B \) and vice versa. The definition of “subset” defines \( A \subseteq B \) precisely as the first half of this criterion: every element of \( A \) is also an element of \( B \). Of course the definition also applies if we switch \( A \) and \( B \): that is, \( B \subseteq A \) iff every element of \( B \) is also an element of \( A \). And that, in turn, is exactly the “vice versa” part of extensionality. In other words, extensionality entails that sets are equal iff they are subsets of one another.

**Proposition set.4.** \( A = B \) iff both \( A \subseteq B \) and \( B \subseteq A \).

Now is also a good opportunity to introduce some further bits of helpful notation. In defining when \( A \) is a subset of \( B \) we said that “every element of \( A \) is . . . ” and filled the “. . . ” with “an element of \( B \)”. But this is such a common shape of expression that it will be helpful to introduce some formal notation for it.

**Definition set.5.** \( \forall x \in A \)\( \varphi \) abbreviates \( \forall x (x \in A \rightarrow \varphi) \). Similarly, \( \exists x \in A \)\( \varphi \) abbreviates \( \exists x (x \in A \land \varphi) \).

Using this notation, we can say that \( A \subseteq B \) iff \( \forall x \in A \)\( x \in B \). Now we move on to considering a certain kind of set: the set of all subsets of a given set.

**Definition set.6 (Power Set).** The set consisting of all subsets of a set \( A \) is called the *power set* of \( A \), written \( \wp(A) \).

\[
\wp(A) = \{ B : B \subseteq A \}
\]
Example set.7. What are all the possible subsets of \( \{a, b, c\} \)? They are: \( \emptyset \), \( \{a\} \), \( \{b\} \), \( \{c\} \), \( \{a, b\} \), \( \{a, c\} \), \( \{b, c\} \), \( \{a, b, c\} \). The set of all these subsets is \( \mathcal{P}(\{a, b, c\}) \):

\[
\mathcal{P}(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}
\]

Problem set.1. List all subsets of \( \{a, b, c, d\} \).

Problem set.2. Show that if \( A \) has \( n \) elements, then \( \mathcal{P}(A) \) has \( 2^n \) elements.

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Bibliography