

set.1 Subsets and Power Sets

sfr:set:sub: sec We will often want to compare sets. And one obvious kind of comparison explanation one might make is as follows: *everything in one set is in the other too*. This situation is sufficiently important for us to introduce some new notation.

Definition set.1 (Subset). If every **element** of a set A is also **an element** of B , then we say that A is a *subset* of B , and write $A \subseteq B$. If A is not a subset of B we write $A \not\subseteq B$. If $A \subseteq B$ but $A \neq B$, we write $A \subsetneq B$ and say that A is a *proper subset* of B .

Example set.2. Every set is a subset of itself, and \emptyset is a subset of every set. The set of even numbers is a subset of the set of natural numbers. Also, $\{a, b\} \subseteq \{a, b, c\}$. But $\{a, b, e\}$ is not a subset of $\{a, b, c\}$.

Example set.3. The number 2 is an **element** of the set of integers, whereas the set of even numbers is a subset of the set of integers. However, a set may happen to *both* be **an element** and a subset of some other set, e.g., $\{0\} \in \{0, \{0\}\}$ and also $\{0\} \subseteq \{0, \{0\}\}$.

Extensionality gives a criterion of identity for sets: $A = B$ iff every **element** of A is also **an element** of B and vice versa. The definition of “subset” defines $A \subseteq B$ precisely as the first half of this criterion: every **element** of A is also **an element** of B . Of course the definition also applies if we switch A and B : that is, $B \subseteq A$ iff every **element** of B is also **an element** of A . And that, in turn, is exactly the “vice versa” part of extensionality. In other words, extensionality entails that sets are equal iff they are subsets of one another.

Proposition set.4. $A = B$ iff both $A \subseteq B$ and $B \subseteq A$.

Now is also a good opportunity to introduce some further bits of helpful notation. In defining when A is a subset of B we said that “every **element** of A is ...,” and filled the “...” with “**an element** of B ”. But this is such a common *shape* of expression that it will be helpful to introduce some formal notation for it.

sfr:set:sub: forallxina **Definition set.5.** $(\forall x \in A)\varphi$ abbreviates $\forall x(x \in A \rightarrow \varphi)$. Similarly, $(\exists x \in A)\varphi$ abbreviates $\exists x(x \in A \wedge \varphi)$.

Using this notation, we can say that $A \subseteq B$ iff $(\forall x \in A)x \in B$.

Now we move on to considering a certain kind of set: the set of all subsets of a given set.

Definition set.6 (Power Set). The set consisting of all subsets of a set A is called the *power set* of A , written $\wp(A)$.

$$\wp(A) = \{B : B \subseteq A\}$$

Example set.7. What are all the possible subsets of $\{a, b, c\}$? They are: \emptyset , $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{a, b, c\}$. The set of all these subsets is $\wp(\{a, b, c\})$:

$$\wp(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$$

Problem set.1. List all subsets of $\{a, b, c, d\}$.

Problem set.2. Show that if A has n elements, then $\wp(A)$ has 2^n elements.

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Bibliography