Subsets and Power Sets

We will often want to compare sets. And one obvious kind of comparison one might make is as follows: *everything in one set is in the other too.* This situation is sufficiently important for us to introduce some new notation.

**Definition set.1 (Subset).** If every element of a set $A$ is also an element of $B$, then we say that $A$ is a *subset* of $B$, and write $A \subseteq B$. If $A$ is not a subset of $B$ we write $A \nsubseteq B$. If $A \subseteq B$ but $A \neq B$, we write $A \subset B$ and say that $A$ is a *proper subset* of $B$.

**Example set.2.** Every set is a subset of itself, and $\emptyset$ is a subset of every set. The set of even numbers is a subset of the set of natural numbers. Also, $\{a, b\} \subseteq \{a, b, c\}$. But $\{a, b, e\}$ is not a subset of $\{a, b, c\}$.

**Example set.3.** The number 2 is an element of the set of integers, whereas the set of even numbers is a subset of the set of integers. However, a set may happen to both be an element and a subset of some other set, e.g., $\{0\} \in \{0, \emptyset\}$ and also $\{0\} \subseteq \{0, \emptyset\}$.

Extensionality gives a criterion of identity for sets: $A = B$ iff every element of $A$ is also an element of $B$ and vice versa. The definition of “subset” defines $A \subseteq B$ precisely as the first half of this criterion: every element of $A$ is also an element of $B$. Of course the definition also applies if we switch $A$ and $B$: that is, $B \subseteq A$ iff every element of $B$ is also an element of $A$. And that, in turn, is exactly the “vice versa” part of extensionality. In other words, extensionality entails that sets are equal iff they are subsets of one another.

**Proposition set.4.** $A = B$ iff both $A \subseteq B$ and $B \subseteq A$.

Now is also a good opportunity to introduce some further bits of helpful notation. In defining when $A$ is a subset of $B$ we said that “every element of $A$ is . . .” and filled the “. . .” with “an element of $B$”. But this is such a common shape of expression that it will be helpful to introduce some formal notation for it.

**Definition set.5.** $(\forall x \in A) \varphi$ abbreviates $\forall x (x \in A \to \varphi)$. Similarly, $(\exists x \in A) \varphi$ abbreviates $\exists x (x \in A \land \varphi)$.

Using this notation, we can say that $A \subseteq B$ iff $(\forall x \in A) x \in B$.

Now we move on to considering a certain kind of set: the set of all subsets of a given set.

**Definition set.6 (Power Set).** The set consisting of all subsets of a set $A$ is called the *power set of* $A$, written $\wp(A)$.

$$\wp(A) = \{ B : B \subseteq A \}$$
Example set.7. What are all the possible subsets of \{a, b, c\}? They are: \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}. The set of all these subsets is \(\wp(\{a, b, c\})\):

\[
\wp(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}
\]

Problem set.1. List all subsets of \{a, b, c, d\}.

Problem set.2. Show that if \(A\) has \(n\) elements, then \(\wp(A)\) has \(2^n\) elements.

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Bibliography