

# Chapter udf

## Sets

### set.1 Basics

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Sets are the most fundamental building blocks of mathematical objects. In fact, almost every mathematical object can be seen as a set of some kind. In logic, as in other parts of mathematics, sets and set-theoretical talk is ubiquitous. So it will be important to discuss what sets are, and introduce the notations necessary to talk about sets and operations on sets in a standard way.

explanation

**Definition set.1** (Set). A *set* is a collection of objects, considered independently of the way it is specified, of the order of the objects in the set, or of their multiplicity. The objects making up the set are called *elements* or *members* of the set. If  $a$  is an **element** of a set  $X$ , we write  $a \in X$  (otherwise,  $a \notin X$ ). The set which has no **elements** is called the *empty* set and denoted by the symbol  $\emptyset$ .

**Example set.2.** Whenever you have a bunch of objects, you can collect them together in a set. The set of Richard's siblings, for instance, is a set that contains one person, and we could write it as  $S = \{\text{Ruth}\}$ . In general, when we have some objects  $a_1, \dots, a_n$ , then the set consisting of exactly those objects is written  $\{a_1, \dots, a_n\}$ . Frequently we'll specify a set by some property that its **elements** share—as we just did, for instance, by specifying  $S$  as the set of Richard's siblings. We'll use the following shorthand notation for that:  $\{x : \dots x \dots\}$ , where the  $\dots x \dots$  stands for the property that  $x$  has to have in order to be counted among the **elements** of the set. In our example, we could have specified  $S$  also as

$$S = \{x : x \text{ is a sibling of Richard}\}.$$

When we say that sets are independent of the way they are specified, we mean that the **elements** of a set are all that matters. For instance, it so happens

explanation

that

$$\begin{aligned} &\{\text{Nicole, Jacob}\}, \\ &\{x : \text{is a niece or nephew of Richard}\}, \text{ and} \\ &\{x : \text{is a child of Ruth}\} \end{aligned}$$

are three ways of specifying one and the same set.

Saying that sets are considered independently of the order of their **elements** and their multiplicity is a fancy way of saying that

$$\begin{aligned} &\{\text{Nicole, Jacob}\} \text{ and} \\ &\{\text{Jacob, Nicole}\} \end{aligned}$$

are two ways of specifying the same set; and that

$$\begin{aligned} &\{\text{Nicole, Jacob}\} \text{ and} \\ &\{\text{Jacob, Nicole, Nicole}\} \end{aligned}$$

are also two ways of specifying the same set. In other words, all that matters is which **elements** a set has. The **elements** of a set are not ordered and each **element** occurs only once. When we *specify* or *describe* a set, **elements** may occur multiple times and in different orders, but any descriptions that only differ in the order of **elements** or in how many times **elements** are listed describes the same set.

**Definition set.3** (Extensionality). If  $X$  and  $Y$  are sets, then  $X$  and  $Y$  are *identical*,  $X = Y$ , iff every **element** of  $X$  is also an **element** of  $Y$ , and vice versa.

explanation

Extensionality gives us a way for showing that sets are identical: to show that  $X = Y$ , show that whenever  $x \in X$  then also  $x \in Y$ , and whenever  $y \in Y$  then also  $y \in X$ .

**Problem set.1.** Show that there is only one empty set, i.e., show that if  $X$  and  $Y$  are sets without members, then  $X = Y$ .

## set.2 Some Important Sets

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**Example set.4.** Mostly we'll be dealing with sets that have mathematical objects as members. You will remember the various sets of numbers:  $\mathbb{N}$  is the set of *natural* numbers  $\{0, 1, 2, 3, \dots\}$ ;  $\mathbb{Z}$  the set of *integers*,

$$\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\};$$

$\mathbb{Q}$  the set of *rational* numbers ( $\mathbb{Q} = \{z/n : z \in \mathbb{Z}, n \in \mathbb{N}, n \neq 0\}$ ); and  $\mathbb{R}$  the set of *real* numbers. These are all *infinite* sets, that is, they each have infinitely many **elements**. As it turns out,  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$  have the same number

of **elements**, while  $\mathbb{R}$  has a whole bunch more— $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$  are “**enumerable** and infinite” whereas  $\mathbb{R}$  is “**non-enumerable**”.

We’ll sometimes also use the set of positive integers  $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$  and the set containing just the first two natural numbers  $\mathbb{B} = \{0, 1\}$ .

**Example set.5** (Strings). Another interesting example is the set  $A^*$  of *finite strings* over an alphabet  $A$ : any finite sequence of elements of  $A$  is a string over  $A$ . We include the *empty string*  $\Lambda$  among the strings over  $A$ , for every alphabet  $A$ . For instance,

$$\mathbb{B}^* = \{\Lambda, 0, 1, 00, 01, 10, 11, \\ 000, 001, 010, 011, 100, 101, 110, 111, 0000, \dots\}.$$

If  $x = x_1 \dots x_n \in A^*$  is a string consisting of  $n$  “letters” from  $A$ , then we say *length* of the string is  $n$  and write  $\text{len}(x) = n$ .

**Example set.6** (Infinite sequences). For any set  $A$  we may also consider the set  $A^\omega$  of infinite sequences of **elements** of  $A$ . An infinite sequence  $a_1 a_2 a_3 a_4 \dots$  consists of a one-way infinite list of objects, each one of which is **an element** of  $A$ .

### set.3 Subsets

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Sets are made up of their elements, and every element of a set is a part of that set. But there is also a sense that some of the elements of a set *taken together* are a “part of” that set. For instance, the number 2 is part of the set of integers, but the set of even numbers is also a part of the set of integers. It’s important to keep those two senses of being part of a set separate.

explanation

**Definition set.7** (Subset). If every element of a set  $X$  is also an element of  $Y$ , then we say that  $X$  is a *subset* of  $Y$ , and write  $X \subseteq Y$ .

**Example set.8.** First of all, every set is a subset of itself, and  $\emptyset$  is a subset of every set. The set of even numbers is a subset of the set of natural numbers. Also,  $\{a, b\} \subseteq \{a, b, c\}$ .

But  $\{a, b, e\}$  is not a subset of  $\{a, b, c\}$ .

Note that a set may contain other sets, not just as subsets but as **elements**! In particular, a set may happen to *both* be **an element** and a subset of another, e.g.,  $\{0\} \in \{0, \{0\}\}$  and also  $\{0\} \subseteq \{0, \{0\}\}$ .

explanation

Extensionality gives a criterion of identity for sets:  $X = Y$  iff every **element** of  $X$  is also **an element** of  $Y$  and vice versa. The definition of “subset” defines  $X \subseteq Y$  precisely as the first half of this criterion: every **element** of  $X$  is also **an element** of  $Y$ . Of course the definition also applies if we switch  $X$  and  $Y$ :  $Y \subseteq X$  iff every **element** of  $Y$  is also **an element** of  $X$ . And that, in turn, is exactly the “vice versa” part of extensionality. In other words, extensionality amounts to:  $X = Y$  iff  $X \subseteq Y$  and  $Y \subseteq X$ .

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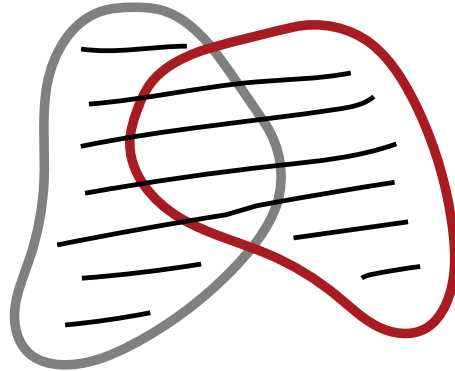


Figure set.1: The union  $X \cup Y$  of two sets is set of **elements** of  $X$  together with those of  $Y$ .

**Definition set.9** (Power Set). The set consisting of all subsets of a set  $X$  is called the *power set of  $X$* , written  $\wp(X)$ .

$$\wp(X) = \{Y : Y \subseteq X\}$$

**Example set.10.** What are all the possible subsets of  $\{a, b, c\}$ ? They are:  $\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$ . The set of all these subsets is  $\wp(\{a, b, c\})$ :

$$\wp(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$$

**Problem set.2.** List all subsets of  $\{a, b, c, d\}$ .

**Problem set.3.** Show that if  $X$  has  $n$  **elements**, then  $\wp(X)$  has  $2^n$  **elements**.

## set.4 Unions and Intersections

explanation

We can define new sets by abstraction, and the property used to define the new set can mention sets we've already defined. So for instance, if  $X$  and  $Y$  are sets, the set  $\{x : x \in X \vee x \in Y\}$  defines a set which consists of all those objects which are **elements** of either  $X$  or  $Y$ , i.e., it's the set that combines the **elements** of  $X$  and  $Y$ . This operation on sets—combining them—is very useful and common, and so we give it a name and a symbol.

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**Definition set.11** (Union). The *union* of two sets  $X$  and  $Y$ , written  $X \cup Y$ , is the set of all things which are **elements** of  $X$ ,  $Y$ , or both.

$$X \cup Y = \{x : x \in X \vee x \in Y\}$$

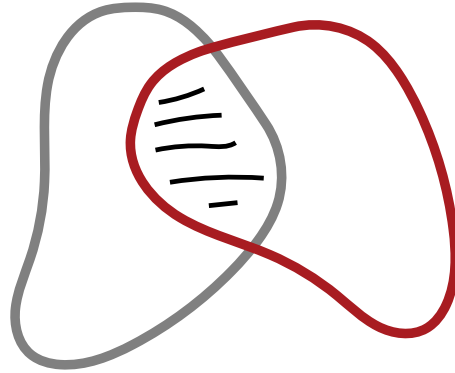


Figure set.2: The intersection  $X \cap Y$  of two sets is the set of **elements** they have in common.

**Example set.12.** Since the multiplicity of **elements** doesn't matter, the union of two sets which have **an element** in common contains that **element** only once, e.g.,  $\{a, b, c\} \cup \{a, 0, 1\} = \{a, b, c, 0, 1\}$ .

The union of a set and one of its subsets is just the bigger set:  $\{a, b, c\} \cup \{a\} = \{a, b, c\}$ .

The union of a set with the empty set is identical to the set:  $\{a, b, c\} \cup \emptyset = \{a, b, c\}$ .

**Problem set.4.** Prove rigorously that if  $X \subseteq Y$ , then  $X \cup Y = Y$ .

The operation that forms the set of all **elements** that  $X$  and  $Y$  have in [common](#) is called their *intersection*.

**Definition set.13** (Intersection). The *intersection* of two sets  $X$  and  $Y$ , written  $X \cap Y$ , is the set of all things which are **elements** of both  $X$  and  $Y$ .

$$X \cap Y = \{x : x \in X \wedge x \in Y\}$$

Two sets are called *disjoint* if their intersection is empty. This means they have no **elements** in common.

**Example set.14.** If two sets have no **elements** in common, their intersection is empty:  $\{a, b, c\} \cap \{0, 1\} = \emptyset$ .

If two sets do have **elements** in common, their intersection is the set of all those:  $\{a, b, c\} \cap \{a, b, d\} = \{a, b\}$ .

The intersection of a set with one of its subsets is just the smaller set:  $\{a, b, c\} \cap \{a, b\} = \{a, b\}$ .

The intersection of any set with the empty set is empty:  $\{a, b, c\} \cap \emptyset = \emptyset$ .

**Problem set.5.** Prove rigorously that if  $X \subseteq Y$ , then  $X \cap Y = X$ .

explanation

We can also form the union or intersection of more than two sets. An elegant way of dealing with this in general is the following: suppose you collect all the sets you want to form the union (or intersection) of into a single set. Then we can define the union of all our original sets as the set of all objects which belong to at least one **element** of the set, and the intersection as the set of all objects which belong to every **element** of the set.

**Definition set.15.** If  $Z$  is a set of sets, then  $\bigcup Z$  is the set of **elements** of **elements** of  $Z$ :

$$\begin{aligned}\bigcup Z &= \{x : x \text{ belongs to an element of } Z\}, \text{ i.e.,} \\ \bigcup Z &= \{x : \text{there is a } Y \in Z \text{ so that } x \in Y\}\end{aligned}$$

**Definition set.16.** If  $Z$  is a set of sets, then  $\bigcap Z$  is the set of objects which all elements of  $Z$  have in common:

$$\begin{aligned}\bigcap Z &= \{x : x \text{ belongs to every element of } Z\}, \text{ i.e.,} \\ \bigcap Z &= \{x : \text{for all } Y \in Z, x \in Y\}\end{aligned}$$

**Example set.17.** Suppose  $Z = \{\{a, b\}, \{a, d, e\}, \{a, d\}\}$ . Then  $\bigcup Z = \{a, b, d, e\}$  and  $\bigcap Z = \{a\}$ .

We could also do the same for a sequence of sets  $X_1, X_2, \dots$

$$\begin{aligned}\bigcup_i X_i &= \{x : x \text{ belongs to one of the } X_i\} \\ \bigcap_i X_i &= \{x : x \text{ belongs to every } X_i\}.\end{aligned}$$

**Definition set.18** (Difference). The *difference*  $X \setminus Y$  is the set of all **elements** of  $X$  which are not also **elements** of  $Y$ , i.e.,

$$X \setminus Y = \{x : x \in X \text{ and } x \notin Y\}.$$

## set.5 Pairs, Tuples, Cartesian Products

explanation

Sets have no order to their elements. We just think of them as an unordered collection. So if we want to represent order, we use *ordered pairs*  $\langle x, y \rangle$ . In an unordered pair  $\{x, y\}$ , the order does not matter:  $\{x, y\} = \{y, x\}$ . In an ordered pair, it does: if  $x \neq y$ , then  $\langle x, y \rangle \neq \langle y, x \rangle$ .

Sometimes we also want ordered sequences of more than two objects, e.g., *triples*  $\langle x, y, z \rangle$ , *quadruples*  $\langle x, y, z, u \rangle$ , and so on. In fact, we can think of triples as special ordered pairs, where the first element is itself an ordered pair:  $\langle x, y, z \rangle$  is short for  $\langle \langle x, y \rangle, z \rangle$ . The same is true for quadruples:  $\langle x, y, z, u \rangle$  is short for  $\langle \langle \langle x, y \rangle, z \rangle, u \rangle$ , and so on. In general, we talk of *ordered  $n$ -tuples*  $\langle x_1, \dots, x_n \rangle$ .

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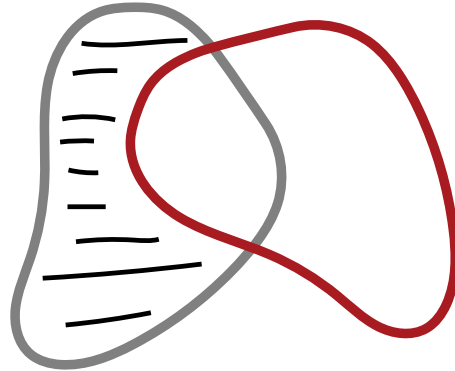


Figure set.3: The difference  $X \setminus Y$  of two sets is the set of those **elements** of  $X$  which are not also elements of  $Y$ .

**Definition set.19** (Cartesian product). Given sets  $X$  and  $Y$ , their *Cartesian product*  $X \times Y$  is  $\{\langle x, y \rangle : x \in X \text{ and } y \in Y\}$ .

**Example set.20.** If  $X = \{0, 1\}$ , and  $Y = \{1, a, b\}$ , then their product is

$$X \times Y = \{\langle 0, 1 \rangle, \langle 0, a \rangle, \langle 0, b \rangle, \langle 1, 1 \rangle, \langle 1, a \rangle, \langle 1, b \rangle\}.$$

**Example set.21.** If  $X$  is a set, the product of  $X$  with itself,  $X \times X$ , is also written  $X^2$ . It is the set of *all* pairs  $\langle x, y \rangle$  with  $x, y \in X$ . The set of all triples  $\langle x, y, z \rangle$  is  $X^3$ , and so on. We can give an inductive definition:

$$\begin{aligned} X^1 &= X \\ X^{k+1} &= X^k \times X \end{aligned}$$

**Problem set.6.** List all **elements** of  $\{1, 2, 3\}^3$ .

**Proposition set.22.** If  $X$  has  $n$  **elements** and  $Y$  has  $m$  **elements**, then  $X \times Y$  has  $n \cdot m$  **elements**.

*Proof.* For every **element**  $x$  in  $X$ , there are  $m$  **elements** of the form  $\langle x, y \rangle \in X \times Y$ . Let  $Y_x = \{\langle x, y \rangle : y \in Y\}$ . Since whenever  $x_1 \neq x_2$ ,  $\langle x_1, y \rangle \neq \langle x_2, y \rangle$ ,  $Y_{x_1} \cap Y_{x_2} = \emptyset$ . But if  $X = \{x_1, \dots, x_n\}$ , then  $X \times Y = Y_{x_1} \cup \dots \cup Y_{x_n}$ , and so has  $n \cdot m$  **elements**.

To visualize this, arrange the **elements** of  $X \times Y$  in a grid:

$$\begin{aligned} Y_{x_1} &= \{\langle x_1, y_1 \rangle & \langle x_1, y_2 \rangle & \dots & \langle x_1, y_m \rangle\} \\ Y_{x_2} &= \{\langle x_2, y_1 \rangle & \langle x_2, y_2 \rangle & \dots & \langle x_2, y_m \rangle\} \\ &\vdots & & & \vdots \\ Y_{x_n} &= \{\langle x_n, y_1 \rangle & \langle x_n, y_2 \rangle & \dots & \langle x_n, y_m \rangle\} \end{aligned}$$

Since the  $x_i$  are all different, and the  $y_j$  are all different, no two of the pairs in this grid are the same, and there are  $n \cdot m$  of them.  $\square$

**Problem set.7.** Show, by induction on  $k$ , that for all  $k \geq 1$ , if  $X$  has  $n$  elements, then  $X^k$  has  $n^k$  elements.

**Example set.23.** If  $X$  is a set, a *word* over  $X$  is any sequence of elements of  $X$ . A sequence can be thought of as an  $n$ -tuple of elements of  $X$ . For instance, if  $X = \{a, b, c\}$ , then the sequence “ $bac$ ” can be thought of as the triple  $\langle b, a, c \rangle$ . Words, i.e., sequences of symbols, are of crucial importance in computer science, of course. By convention, we count elements of  $X$  as sequences of length 1, and  $\emptyset$  as the sequence of length 0. The set of *all* words over  $X$  then is

$$X^* = \{\emptyset\} \cup X \cup X^2 \cup X^3 \cup \dots$$

## set.6 Russell’s Paradox

We said that one can define sets by specifying a property that its elements share, e.g., defining the set of Richard’s siblings as

$$S = \{x : x \text{ is a sibling of Richard}\}.$$

In the very general context of mathematics one must be careful, however: not every property lends itself to *comprehension*. Some properties do not define sets. If they did, we would run into outright contradictions. One example of such a case is Russell’s Paradox.

Sets may be elements of other sets—for instance, the power set of a set  $X$  is made up of sets. And so it makes sense, of course, to ask or investigate whether a set is an element of another set. Can a set be a member of itself? Nothing about the idea of a set seems to rule this out. For instance, surely *all* sets form a collection of objects, so we should be able to collect them into a single set—the set of all sets. And it, being a set, would be an element of the set of all sets.

Russell’s Paradox arises when we consider the property of not having itself as an element. The set of all sets does not have this property, but all sets we have encountered so far have it.  $\mathbb{N}$  is not an element of  $\mathbb{N}$ , since it is a set, not a natural number.  $\wp(X)$  is generally not an element of  $\wp(X)$ ; e.g.,  $\wp(\mathbb{R}) \notin \wp(\mathbb{R})$  since it is a set of sets of real numbers, not a set of real numbers. What if we suppose that there is a set of all sets that do not have themselves as an element? Does

$$R = \{x : x \notin x\}$$

exist?

If  $R$  exists, it makes sense to ask if  $R \in R$  or not—it must be either  $\in R$  or  $\notin R$ . Suppose the former is true, i.e.,  $R \in R$ .  $R$  was defined as the set of all sets that are not elements of themselves, and so if  $R \in R$ , then  $R$  does not have this defining property of  $R$ . But only sets that have this property are in  $R$ , hence,  $R$  cannot be an element of  $R$ , i.e.,  $R \notin R$ . But  $R$  can’t both be and not be an element of  $R$ , so we have a contradiction.



Since the assumption that  $R \in R$  leads to a contradiction, we have  $R \notin R$ . But this also leads to a contradiction! For if  $R \notin R$ , it does have the defining property of  $R$ , and so would be an element of  $R$  just like all the other non-self-containing sets. And again, it can't both not be and be an element of  $R$ .

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# Bibliography