

set.1 Russell's Paradox

sfr:set:rus:
sec Extensionality licenses the notation $\{x : \varphi(x)\}$, for *the* set of x 's such that $\varphi(x)$. However, all that extensionality *really* licenses is the following thought. *If* there is a set whose members are all and only the φ 's, *then* there is only one such set. Otherwise put: having fixed some φ , the set $\{x : \varphi(x)\}$ is unique, *if it exists*.

But this conditional is important! Crucially, not every property lends itself to *comprehension*. That is, some properties do *not* define sets. If they all did, then we would run into outright contradictions. The most famous example of this is Russell's Paradox.

Sets may be **elements** of other sets—for instance, the power set of a set A is made up of sets. And so it makes sense to ask or investigate whether a set is **an element** of another set. Can a set be a member of itself? Nothing about the idea of a set seems to rule this out. For instance, if *all* sets form a collection of objects, one might think that they can be collected into a single set—the set of all sets. And it, being a set, would be **an element** of the set of all sets.

Russell's Paradox arises when we consider the property of not having itself as **an element**, of being *non-self-membered*. What if we suppose that there is a set of all sets that do not have themselves as **an element**? Does

$$R = \{x : x \notin x\}$$

exist? It turns out that we can prove that it does not.

sfr:set:rus:
thm:russells-paradox **Theorem set.1 (Russell's Paradox).** *There is no set $R = \{x : x \notin x\}$.*

Proof. For reductio, suppose that $R = \{x : x \notin x\}$ exists. Then $R \in R$ iff $R \notin R$, since sets are extensional. But this is a contradiction. \square

Let's run through the proof that no set R of non-self-membered sets can exist more slowly. If R exists, it makes sense to ask if $R \in R$ or not—it must be either $\in R$ or $\notin R$. Suppose the former is true, i.e., $R \in R$. R was defined as the set of all sets that are not **elements** of themselves, and so if $R \in R$, then R does not have this defining property of R . But only sets that have this property are in R , hence, R cannot be **an element** of R , i.e., $R \notin R$. But R can't both be and not be **an element** of R , so we have a contradiction. explanation

Since the assumption that $R \in R$ leads to a contradiction, we have $R \notin R$. But this also leads to a contradiction! For if $R \notin R$, it does have the defining property of R , and so would be **an element** of R just like all the other non-self-membered sets. And again, it can't both not be and be **an element** of R .

How do we set up a set theory which avoids falling into Russell's Paradox, i.e., which avoids making the *inconsistent* claim that $R = \{x : x \notin x\}$ exists? Well, we would need to lay down axioms which give us very precise conditions for stating when sets exist (and when they don't). digression

The set theory sketched in this chapter doesn't do this. It's *genuinely naïve*. It tells you only that sets obey extensionality and that, if you have some sets,

you can form their union, intersection, etc. It is possible to develop set theory more rigorously than this.

Photo Credits

Bibliography