

set.1 Russell's Paradox

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sec We said that one can define sets by specifying a property that its **elements** share, e.g., defining the set of Richard's siblings as

$$S = \{x : x \text{ is a sibling of Richard}\}.$$

In the very general context of mathematics one must be careful, however: not every property lends itself to *comprehension*. Some properties do not define sets. If they did, we would run into outright contradictions. One example of such a case is Russell's Paradox.

Sets may be **elements** of other sets—for instance, the power set of a set X is made up of sets. And so it makes sense, of course, to ask or investigate whether a set is **an element** of another set. Can a set be a member of itself? Nothing about the idea of a set seems to rule this out. For instance, surely *all* sets form a collection of objects, so we should be able to collect them into a single set—the set of all sets. And it, being a set, would be **an element** of the set of all sets.

Russell's Paradox arises when we consider the property of not having itself as **an element**. The set of all sets does not have this property, but all sets we have encountered so far have it. \mathbb{N} is not **an element** of \mathbb{N} , since it is a set, not a natural number. $\wp(X)$ is generally not **an element** of $\wp(X)$; e.g., $\wp(\mathbb{R}) \notin \wp(\mathbb{R})$ since it is a set of sets of real numbers, not a set of real numbers. What if we suppose that there is a set of all sets that do not have themselves as **an element**? Does

$$R = \{x : x \notin x\}$$

exist?

If R exists, it makes sense to ask if $R \in R$ or not—it must be either $\in R$ or $\notin R$. Suppose the former is true, i.e., $R \in R$. R was defined as the set of all sets that are not **elements** of themselves, and so if $R \in R$, then R does not have this defining property of R . But only sets that have this property are in R , hence, R cannot be **an element** of R , i.e., $R \notin R$. But R can't both be and not be **an element** of R , so we have a contradiction.

Since the assumption that $R \in R$ leads to a contradiction, we have $R \notin R$. But this also leads to a contradiction! For if $R \notin R$, it does have the defining property of R , and so would be **an element** of R just like all the other non-self-containing sets. And again, it can't both not be and be **an element** of R .

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Bibliography