set.1 Russell’s Paradox

We said that one can define sets by specifying a property that its elements share, e.g., defining the set of Richard’s siblings as

\[ S = \{ x : x \text{ is a sibling of Richard} \}. \]

In the very general context of mathematics one must be careful, however: not every property lends itself to comprehension. Some properties do not define sets. If they did, we would run into outright contradictions. One example of such a case is Russell’s Paradox.

Sets may be elements of other sets—for instance, the power set of a set \( X \) is made up of sets. And so it makes sense, of course, to ask or investigate whether a set is an element of another set. Can a set be a member of itself? Nothing about the idea of a set seems to rule this out. For instance, surely all sets form a collection of objects, so we should be able to collect them into a single set—the set of all sets. And it, being a set, would be an element of the set of all sets.

Russell’s Paradox arises when we consider the property of not having itself as an element. The set of all sets does not have this property, but all sets we have encountered so far have it. \( \mathbb{N} \) is not an element of \( \mathbb{N} \), since it is a set, not a natural number. \( \wp(X) \) is generally not an element of \( \wp(X) \); e.g., \( \wp(\mathbb{R}) \notin \wp(\mathbb{R}) \) since it is a set of sets of real numbers, not a set of real numbers. What if we suppose that there is a set of all sets that do not have themselves as an element? Does

\[ R = \{ x : x \notin x \} \]

exist?

If \( R \) exists, it makes sense to ask if \( R \in R \) or not—it must be either \( \in R \) or \( \notin R \). Suppose the former is true, i.e., \( R \in R \). \( R \) was defined as the set of all sets that are not elements of themselves, and so if \( R \in R \), then \( R \) does not have this defining property of \( R \). But only sets that have this property are in \( R \), hence, \( R \) cannot be an element of \( R \), i.e., \( R \notin R \). But \( R \) can’t both be and not be an element of \( R \), so we have a contradiction.

Since the assumption that \( R \in R \) leads to a contradiction, we have \( R \notin R \). But this also leads to a contradiction! For if \( R \notin R \), it does have the defining property of \( R \), and so would be an element of \( R \) just like all the other non-self-containing sets. And again, it can’t both not be and be an element of \( R \).

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Bibliography