

## set.1 Pairs, Tuples, Cartesian Products

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Sets have no order to their elements. We just think of them as an unordered explanation collection. So if we want to represent order, we use *ordered pairs*  $\langle x, y \rangle$ . In an unordered pair  $\{x, y\}$ , the order does not matter:  $\{x, y\} = \{y, x\}$ . In an ordered pair, it does: if  $x \neq y$ , then  $\langle x, y \rangle \neq \langle y, x \rangle$ .

Sometimes we also want ordered sequences of more than two objects, e.g., *triples*  $\langle x, y, z \rangle$ , *quadruples*  $\langle x, y, z, u \rangle$ , and so on. In fact, we can think of triples as special ordered pairs, where the first element is itself an ordered pair:  $\langle x, y, z \rangle$  is short for  $\langle \langle x, y \rangle, z \rangle$ . The same is true for quadruples:  $\langle x, y, z, u \rangle$  is short for  $\langle \langle \langle x, y \rangle, z \rangle, u \rangle$ , and so on. In general, we talk of *ordered  $n$ -tuples*  $\langle x_1, \dots, x_n \rangle$ .

**Definition set.1** (Cartesian product). Given sets  $X$  and  $Y$ , their *Cartesian product*  $X \times Y$  is  $\{\langle x, y \rangle : x \in X \text{ and } y \in Y\}$ .

**Example set.2.** If  $X = \{0, 1\}$ , and  $Y = \{1, a, b\}$ , then their product is

$$X \times Y = \{\langle 0, 1 \rangle, \langle 0, a \rangle, \langle 0, b \rangle, \langle 1, 1 \rangle, \langle 1, a \rangle, \langle 1, b \rangle\}.$$

**Example set.3.** If  $X$  is a set, the product of  $X$  with itself,  $X \times X$ , is also written  $X^2$ . It is the set of *all* pairs  $\langle x, y \rangle$  with  $x, y \in X$ . The set of all triples  $\langle x, y, z \rangle$  is  $X^3$ , and so on. We can give an inductive definition:

$$\begin{aligned} X^1 &= X \\ X^{k+1} &= X^k \times X \end{aligned}$$

**Problem set.1.** List all elements of  $\{1, 2, 3\}^3$ .

**Proposition set.4.** If  $X$  has  $n$  elements and  $Y$  has  $m$  elements, then  $X \times Y$  has  $n \cdot m$  elements.

*Proof.* For every element  $x$  in  $X$ , there are  $m$  elements of the form  $\langle x, y \rangle \in X \times Y$ . Let  $Y_x = \{\langle x, y \rangle : y \in Y\}$ . Since whenever  $x_1 \neq x_2$ ,  $\langle x_1, y \rangle \neq \langle x_2, y \rangle$ ,  $Y_{x_1} \cap Y_{x_2} = \emptyset$ . But if  $X = \{x_1, \dots, x_n\}$ , then  $X \times Y = Y_{x_1} \cup \dots \cup Y_{x_n}$ , and so has  $n \cdot m$  elements.

To visualize this, arrange the elements of  $X \times Y$  in a grid:

$$\begin{array}{l} Y_{x_1} = \{ \langle x_1, y_1 \rangle \quad \langle x_1, y_2 \rangle \quad \dots \quad \langle x_1, y_m \rangle \} \\ Y_{x_2} = \{ \langle x_2, y_1 \rangle \quad \langle x_2, y_2 \rangle \quad \dots \quad \langle x_2, y_m \rangle \} \\ \vdots \\ Y_{x_n} = \{ \langle x_n, y_1 \rangle \quad \langle x_n, y_2 \rangle \quad \dots \quad \langle x_n, y_m \rangle \} \end{array}$$

Since the  $x_i$  are all different, and the  $y_j$  are all different, no two of the pairs in this grid are the same, and there are  $n \cdot m$  of them.  $\square$

**Problem set.2.** Show, by induction on  $k$ , that for all  $k \geq 1$ , if  $X$  has  $n$  elements, then  $X^k$  has  $n^k$  elements.

**Example set.5.** If  $X$  is a set, a *word* over  $X$  is any sequence of **elements** of  $X$ . A sequence can be thought of as an  $n$ -tuple of **elements** of  $X$ . For instance, if  $X = \{a, b, c\}$ , then the sequence “ $bac$ ” can be thought of as the triple  $\langle b, a, c \rangle$ . Words, i.e., sequences of symbols, are of crucial importance in computer science, of course. By convention, we count **elements** of  $X$  as sequences of length 1, and  $\emptyset$  as the sequence of length 0. The set of *all* words over  $X$  then is

$$X^* = \{\emptyset\} \cup X \cup X^2 \cup X^3 \cup \dots$$

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## Bibliography