set.1 Pairs, Tuples, Cartesian Products

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Sets have no order to their elements. We just think of them as an unordered collection. So if we want to represent order, we use *ordered pairs* $\langle x, y\rangle$. In an unordered pair $\{x, y\}$, the order does not matter: $\{x, y\} = \{y, x\}$. In an ordered pair, it does: if $x \neq y$, then $\langle x, y\rangle \neq \langle y, x\rangle$. explanation

Sometimes we also want ordered sequences of more than two objects, e.g., *triples* $\langle x, y, z\rangle$, *quadruples* $\langle x, y, z, u\rangle$, and so on. In fact, we can think of triples as special ordered pairs, where the first element is itself an ordered pair: $\langle x, y, z\rangle$ is short for $\langle\langle x, y\rangle, z\rangle$. The same is true for quadruples: $\langle x, y, z, u\rangle$ is short for $\langle\langle\langle x, y\rangle, z\rangle, u\rangle$, and so on. In general, we talk of *ordered n-tuples* $\langle x_1, \dots, x_n\rangle$.

Definition set.1 (Cartesian product)**.** Given sets X and Y, their *Cartesian product* $X \times Y$ is $\{\langle x, y\rangle : x \in X \text{ and } y \in Y\}$.

Example set.2. If $X = \{0, 1\}$, and $Y = \{1, a, b\}$, then their product is

$$X \times Y = \{\langle 0, 1\rangle, \langle 0, a\rangle, \langle 0, b\rangle, \langle 1, 1\rangle, \langle 1, a\rangle, \langle 1, b\rangle\}.$$

Example set.3. If X is a set, the product of X with itself, $X \times X$, is also written X^2. It is the set of *all* pairs $\langle x, y\rangle$ with $x, y \in X$. The set of all triples $\langle x, y, z\rangle$ is X^3, and so on. We can give an inductive definition:

$$\begin{aligned} X^1 &= X \\ X^{k+1} &= X^k \times X \end{aligned}$$

Problem set.1. List all elements of $\{1, 2, 3\}^3$.

Proposition set.4. *If X has n elements and Y has m elements, then $X \times Y$ has $n \cdot m$ elements.*

Proof. For every element x in X, there are m elements of the form $\langle x, y\rangle \in X \times Y$. Let $Y_x = \{\langle x, y\rangle : y \in Y\}$. Since whenever $x_1 \neq x_2$, $\langle x_1, y\rangle \neq \langle x_2, y\rangle$, $Y_{x_1} \cap Y_{x_2} = \emptyset$. But if $X = \{x_1, \dots, x_n\}$, then $X \times Y = Y_{x_1} \cup \dots \cup Y_{x_n}$, and so has $n \cdot m$ elements.

To visualize this, arrange the elements of $X \times Y$ in a grid:

$$\begin{array}{rlllll} Y_{x_1} = & \{\langle x_1, y_1\rangle & \langle x_1, y_2\rangle & \dots & \langle x_1, y_m\rangle\} \\ Y_{x_2} = & \{\langle x_2, y_1\rangle & \langle x_2, y_2\rangle & \dots & \langle x_2, y_m\rangle\} \\ \vdots & & \vdots & & \\ Y_{x_n} = & \{\langle x_n, y_1\rangle & \langle x_n, y_2\rangle & \dots & \langle x_n, y_m\rangle\} \end{array}$$

Since the x_i are all different, and the y_j are all different, no two of the pairs in this grid are the same, and there are $n \cdot m$ of them. □

Problem set.2. Show, by induction on k, that for all $k \geq 1$, if X has n elements, then X^k has n^k elements.

Example set.5. If X is a set, a *word* over X is any sequence of elements of X. A sequence can be thought of as an n-tuple of elements of X. For instance, if $X = \{a, b, c\}$, then the sequence "bac" can be thought of as the triple $\langle b, a, c\rangle$. Words, i.e., sequences of symbols, are of crucial importance in computer science, of course. By convention, we count elements of X as sequences of length 1, and $\emptyset$ as the sequence of length 0. The set of *all* words over X then is

$$X^* = \{\emptyset\} \cup X \cup X^2 \cup X^3 \cup \dots$$

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Bibliography