set.1 Pairs, Tuples, Cartesian Products

sfr:set:pai: sec

It follows from extensionality that sets have no order to their elements. So if we want to represent order, we use *ordered pairs* $\langle x, y \rangle$. In an unordered pair $\{x, y\}$, the order does not matter: $\{x, y\} = \{y, x\}$. In an ordered pair, it does: if $x \neq y$, then $\langle x, y \rangle \neq \langle y, x \rangle$. explanation

How should we think about ordered pairs in set theory? Crucially, we want to preserve the idea that ordered pairs are identical iff they share the same first element and share the same second element, i.e.:

$$\langle a, b \rangle = \langle c, d \rangle \text{ iff both } a = c \text{ and } b = d.$$

We can define ordered pairs in set theory using the Wiener-Kuratowski definition.

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Definition set.1 (Ordered pair). $\langle a, b \rangle = \{\{a\}, \{a, b\}\}$.

Problem set.1. Using Definition set.1, prove that $\langle a, b \rangle = \langle c, d \rangle$ iff both $a = c$ and $b = d$.

Having fixed a definition of an ordered pair, we can use it to define further sets. For example, sometimes we also want ordered sequences of more than two objects, e.g., *triples* $\langle x, y, z \rangle$, *quadruples* $\langle x, y, z, u \rangle$, and so on. We can think of triples as special ordered pairs, where the first element is itself an ordered pair: $\langle x, y, z \rangle$ is $\langle \langle x, y \rangle, z \rangle$. The same is true for quadruples: $\langle x, y, z, u \rangle$ is $\langle \langle \langle x, y \rangle, z \rangle, u \rangle$, and so on. In general, we talk of *ordered n-tuples* $\langle x_1, \dots, x_n \rangle$. explanation

Certain sets of ordered pairs, or other ordered n-tuples, will be useful.

Definition set.2 (Cartesian product). Given sets A and B, their *Cartesian product* $A \times B$ is defined by

$$A \times B = \{\langle x, y \rangle : x \in A \text{ and } y \in B\}.$$

Example set.3. If $A = \{0, 1\}$, and $B = \{1, a, b\}$, then their product is

$$A \times B = \{\langle 0, 1 \rangle, \langle 0, a \rangle, \langle 0, b \rangle, \langle 1, 1 \rangle, \langle 1, a \rangle, \langle 1, b \rangle\}.$$

Example set.4. If A is a set, the product of A with itself, $A \times A$, is also written A^2. It is the set of *all* pairs $\langle x, y \rangle$ with $x, y \in A$. The set of all triples $\langle x, y, z \rangle$ is A^3, and so on. We can give a recursive definition:

$$\begin{aligned} A^1 &= A \\ A^{k+1} &= A^k \times A \end{aligned}$$

Problem set.2. List all elements of $\{1, 2, 3\}^3$.

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Proposition set.5. *If A has n elements and B has m elements, then $A \times B$ has $n \cdot m$ elements.*

Proof. For every element x in A, there are m elements of the form $\langle x, y\rangle \in A \times B$. Let $B_x = \{\langle x, y\rangle : y \in B\}$. Since whenever $x_1 \neq x_2$, $\langle x_1, y\rangle \neq \langle x_2, y\rangle$, $B_{x_1} \cap B_{x_2} = \emptyset$. But if $A = \{x_1, \dots, x_n\}$, then $A \times B = B_{x_1} \cup \dots \cup B_{x_n}$, and so has $n \cdot m$ elements.

To visualize this, arrange the elements of $A \times B$ in a grid:

$$\begin{array}{rcccc} B_{x_1} = & \{\langle x_1, y_1\rangle & \langle x_1, y_2\rangle & \dots & \langle x_1, y_m\rangle\} \\ B_{x_2} = & \{\langle x_2, y_1\rangle & \langle x_2, y_2\rangle & \dots & \langle x_2, y_m\rangle\} \\ \vdots & & \vdots & & \\ B_{x_n} = & \{\langle x_n, y_1\rangle & \langle x_n, y_2\rangle & \dots & \langle x_n, y_m\rangle\} \end{array}$$

Since the x_i are all different, and the y_j are all different, no two of the pairs in this grid are the same, and there are $n \cdot m$ of them. □

Problem set.3. Show, by induction on k, that for all $k \geq 1$, if A has n elements, then A^k has n^k elements.

Example set.6. If A is a set, a *word* over A is any sequence of elements of A. A sequence can be thought of as an n-tuple of elements of A. For instance, if $A = \{a, b, c\}$, then the sequence "*bac*" can be thought of as the triple $\langle b, a, c\rangle$. Words, i.e., sequences of symbols, are of crucial importance in computer science. By convention, we count elements of A as sequences of length 1, and $\emptyset$ as the sequence of length 0. The set of *all* words over A then is

$$A^* = \{\emptyset\} \cup A \cup A^2 \cup A^3 \cup \dots$$

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Bibliography