

set.1 Extensionality

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A *set* is a collection of objects, considered as a single object. The objects making up the set are called *elements* or *members* of the set. If x is an **element** of a set A , we write $x \in A$; if not, we write $x \notin A$. The set which has no **elements** is called the *empty* set and denoted “ \emptyset ”.

It does not matter how we *specify* the set, or how we *order* its **elements**, or indeed how *many times* we count its **elements**. All that matters are what its **elements** are. We codify this in the following principle. explanation

Definition set.1 (Extensionality). If A and B are sets, then $A = B$ iff every **element** of A is also an **element** of B , and vice versa.

Extensionality licenses some notation. In general, when we have some objects a_1, \dots, a_n , then $\{a_1, \dots, a_n\}$ is *the* set whose **elements** are a_1, \dots, a_n . We emphasise the word “*the*”, since extensionality tells us that there can be only *one* such set. Indeed, extensionality also licenses the following:

$$\{a, a, b\} = \{a, b\} = \{b, a\}.$$

This delivers on the point that, when we consider sets, we don’t care about the order of their **elements**, or how many times they are specified.

Example set.2. Whenever you have a bunch of objects, you can collect them together in a set. The set of Richard’s siblings, for instance, is a set that contains one person, and we could write it as $S = \{\text{Ruth}\}$. The set of positive integers less than 4 is $\{1, 2, 3\}$, but it can also be written as $\{3, 2, 1\}$ or even as $\{1, 2, 1, 2, 3\}$. These are all the same set, by extensionality. For every **element** of $\{1, 2, 3\}$ is also an **element** of $\{3, 2, 1\}$ (and of $\{1, 2, 1, 2, 3\}$), and vice versa.

Frequently we’ll specify a set by some property that its **elements** share. We’ll use the following shorthand notation for that: $\{x : \varphi(x)\}$, where the $\varphi(x)$ stands for the property that x has to have in order to be counted among the **elements** of the set.

Example set.3. In our example, we could have specified S also as

$$S = \{x : x \text{ is a sibling of Richard}\}.$$

Example set.4. A number is called *perfect* iff it is equal to the sum of its proper divisors (i.e., numbers that evenly divide it but aren’t identical to the number). For instance, 6 is perfect because its proper divisors are 1, 2, and 3, and $6 = 1 + 2 + 3$. In fact, 6 is the only positive integer less than 10 that is perfect. So, using extensionality, we can say:

$$\{6\} = \{x : x \text{ is perfect and } 0 \leq x \leq 10\}$$

We read the notation on the right as “the set of x ’s such that x is perfect and $0 \leq x \leq 10$ ”. The identity here confirms that, when we consider sets, we don’t care

about how they are specified. And, more generally, extensionality guarantees that there is always only one set of x 's such that $\varphi(x)$. So, extensionality justifies calling $\{x : \varphi(x)\}$ *the* set of x 's such that $\varphi(x)$.

Extensionality gives us a way for showing that sets are identical: to show that $A = B$, show that whenever $x \in A$ then also $x \in B$, and whenever $y \in B$ then also $y \in A$.

Problem set.1. Prove that there is at most one empty set, i.e., show that if A and B are sets without **elements**, then $A = B$.

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Bibliography