

## set.1 Extensionality

sfr:set:bas:  
sec A *set* is a collection of objects, considered as a single object. The objects making up the set are called *elements* or *members* of the set. If  $x$  is an element of a set  $a$ , we write  $x \in a$ ; if not, we write  $x \notin a$ . The set which has no elements is called the *empty* set and denoted “ $\emptyset$ ”.

It does not matter how we *specify* the set, or how we *order* its elements, or indeed how *many times* we count its elements. All that matters are what its elements are. We codify this in the following principle. explanation

**Definition set.1 (Extensionality).** If  $A$  and  $B$  are sets, then  $A = B$  iff every element of  $A$  is also an element of  $B$ , and vice versa.

Extensionality licenses some notation. In general, when we have some objects  $a_1, \dots, a_n$ , then  $\{a_1, \dots, a_n\}$  is the set whose elements are  $a_1, \dots, a_n$ . We emphasise the word “*the*”, since extensionality tells us that there can be only one such set. Indeed, extensionality also licenses the following:

$$\{a, a, b\} = \{a, b\} = \{b, a\}.$$

This delivers on the point that, when we consider sets, we don’t care about the order of their elements, or how many times they are specified.

**Example set.2.** Whenever you have a bunch of objects, you can collect them together in a set. The set of Richard’s siblings, for instance, is a set that contains one person, and we could write it as  $S = \{\text{Ruth}\}$ . The set of positive integers less than 4 is  $\{1, 2, 3\}$ , but it can also be written as  $\{3, 2, 1\}$  or even as  $\{1, 2, 1, 2, 3\}$ . These are all the same set, by extensionality. For every element of  $\{1, 2, 3\}$  is also an element of  $\{3, 2, 1\}$  (and of  $\{1, 2, 1, 2, 3\}$ ), and vice versa.

Frequently we’ll specify a set by some property that its elements share. We’ll use the following shorthand notation for that:  $\{x : \varphi(x)\}$ , where the  $\varphi(x)$  stands for the property that  $x$  has to have in order to be counted among the elements of the set.

**Example set.3.** In our example, we could have specified  $S$  also as

$$S = \{x : x \text{ is a sibling of Richard}\}.$$

**Example set.4.** A number is called *perfect* iff it is equal to the sum of its proper divisors (i.e., numbers that evenly divide it but aren’t identical to the number). For instance, 6 is perfect because its proper divisors are 1, 2, and 3, and  $6 = 1 + 2 + 3$ . In fact, 6 is the only positive integer less than 10 that is perfect. So, using extensionality, we can say:

$$\{6\} = \{x : x \text{ is perfect and } 0 \leq x \leq 10\}$$

We read the notation on the right as “the set of  $x$ ’s such that  $x$  is perfect and  $0 \leq x \leq 10$ ”. The identity here confirms that, when we consider sets, we don’t care

about how they are specified. And, more generally, extensionality guarantees that there is always only one set of  $x$ 's such that  $\varphi(x)$ . So, extensionality justifies calling  $\{x : \varphi(x)\}$  *the* set of  $x$ 's such that  $\varphi(x)$ .

Extensionality gives us a way for showing that sets are identical: to show that  $A = B$ , show that whenever  $x \in A$  then also  $x \in B$ , and whenever  $y \in B$  then also  $y \in A$ .

**Problem set.1.** Prove that there is at most one empty set, i.e., show that if  $A$  and  $B$  are sets without elements, then  $A = B$ .

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## Bibliography