rel.1 Trees

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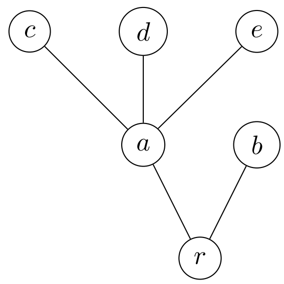
A particular kind of partial order which plays an important role in all parts of logic is a *tree*. Finite trees occur in elementary parts of logic: for example, formulas can be understood in terms of their decomposition into a syntax tree, while derivations in many derivation systems also take the form of finite trees. Infinite trees appear already in the proof of the completeness theorems for propositional and first-order logic, and are used throughout mathematical logic.

The set-theoretic concept of a tree is closely related to the notion of a tree in graph theory. Here is a picture of a (finite) tree:



The lowermost node r is the root. Every node other than r has exactly one parent node immediately below it. We can think of the relation a node x stands in to a node y if y can be reached from x by following edges upwards as x being an *ancestor* of y.

The ancestor relation in a tree is a strict partial order. This motivates the set-theoretic definition. To state it we need two concepts. A *minimal element* in a set A partially ordered by $\leq$ is an element $x \in A$ such that for all $y \in A$ we have that $x \leq y$. A set is *well-ordered* by $\leq$ if every one of its subsets has a minimal element.

Definition rel.1 (Tree). A *tree* is a pair $T = \langle A, \leq \rangle$ such that A is a set and $\leq$ is a partial order on A with a unique minimal element $r \in A$ (called the *root*) such that for all $x \in A$, the set $\{y : y \leq z\}$ is well-ordered by $\leq$.

Definition rel.2 (Successors). Suppose $T = \langle A, \leq \rangle$ is a tree. If $x, y \in A$, $x < y$, and there is no $z \in A$ such that $x < z < y$, then we say that y is a *successor* of x.

The successors of $x \in A$ are also called its *children*. If y is a successor of x, then we call x the *predecessor* or *parent* of y.

Proposition rel.3. *If $\langle A, \leq \rangle$ is a tree, then every $x \in A$ other than the root has at most one predecessor.*

Proof. Suppose $y_1 < x$ and $y_2 < x$ and $y_1 \neq y_2$. Then $\{y_1, y_2\} \subseteq \{z : z < x\}$. Since $\{z : z < x\}$ is well-ordered by $\leq$, its subset $\{y_1, y_2\}$ has a minimal element, which obviously must be either y_1 or y_2. So either $y_1 \leq y_2$ or $y_2 \leq y_1$. We assumed that $y_1 \neq y_2$, so actually either $y_1 < y_2$ or $y_2 < y_1$. Since we assumed that $y_1 < x$ and $y_2 < x$, we furthermore have that either $y_1 < y_2 < x$ or $y_2 < y_1 < x$. So y_1 and y_2 cannot both be predecessors of x. □

Definition rel.4. A tree $T = \langle A, \leq \rangle$ is said to be *infinite* if A is an infinite set, and *finite* otherwise. If T is such that every $x \in A$ has only finitely many successors, then we say that T is *finitely branching*.

Definition rel.5 (Branches). Given a tree $T = \langle A, \leq \rangle$, a *branch* of T is a maximal chain in T, i.e., a set $B \subseteq A$ such that for any $x, y \in B$ either $x \leq y$ or $y \leq x$, and for any $z \in X \setminus B$ there exists $u \in B$ such that neither $z \leq u$ nor $u \leq z$. We use $[T]$ to denote the set of all branches of T.

Example rel.6. A classic example of a finitely branching tree is the *infinite binary tree* of finite sequences of 0s and 1s, sometimes denoted $\{0, 1\}^*$ or $\mathbb{B}^*$, ordered by the extension relation $\sqsubseteq$ (e.g., $101 \sqsubseteq 101101$). Since any binary string can always be extended by adding a 0 or a 1 on the end, this tree contains infinitely many elements: every element s has exactly two successors, $s0$ and $s1$. Its root is the empty sequence Λ.

Example rel.7. Slightly more generally, the set of finite sequences of natural numbers $\mathbb{N}^*$ with the extension relation $\sqsubseteq$ is also a tree. It is obviously not finitely branching: every $s \in \mathbb{N}^*$ has infinitely many successors sn, one for every $n \in \mathbb{N}$. Every $A \subseteq \mathbb{N}^*$ which is closed under $\sqsubseteq$ is a *subtree* of $\mathbb{N}^*$. (That is, A is such that if $s \in A$ and $s' \sqsubseteq s$, then also $s' \in A$.) All finite trees can be represented as finite subtrees of $\mathbb{N}^*$.

Proposition rel.8 (Kőnig's lemma). *If $T = \langle A, \leq \rangle$ is a finitely branching infinite tree, then T has an infinite branch.*

A special case of Kőnig's lemma widely used in computability theory, known as *weak Kőnig's lemma*, is the following: any infinite subtree of $\{0, 1\}^*$ has an infinite branch.

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Bibliography