

# Chapter udf

## Relations

### rel.1 Relations as Sets

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sec You will no doubt remember some interesting relations between objects of explanation some of the sets we've mentioned. For instance, numbers come with an *order relation*  $<$  and from the theory of whole numbers the relation of *divisibility without remainder* (usually written  $n \mid m$ ) may be familiar. There is also the relation *is identical with* that every object bears to itself and to no other thing. But there are many more interesting relations that we'll encounter, and even more possible relations. Before we review them, we'll just point out that we can look at relations as a special sort of set. For this, first recall what a *pair* is: if  $a$  and  $b$  are two objects, we can combine them into the *ordered pair*  $\langle a, b \rangle$ . Note that for ordered pairs the order *does* matter, e.g.,  $\langle a, b \rangle \neq \langle b, a \rangle$ , in contrast to unordered pairs, i.e., 2-element sets, where  $\{a, b\} = \{b, a\}$ .

If  $X$  and  $Y$  are sets, then the *Cartesian product*  $X \times Y$  of  $X$  and  $Y$  is the set of all pairs  $\langle a, b \rangle$  with  $a \in X$  and  $b \in Y$ . In particular,  $X^2 = X \times X$  is the set of all pairs from  $X$ .

Now consider a relation on a set, e.g., the  $<$ -relation on the set  $\mathbb{N}$  of natural numbers, and consider the set of all pairs of numbers  $\langle n, m \rangle$  where  $n < m$ , i.e.,

$$R = \{\langle n, m \rangle : n, m \in \mathbb{N} \text{ and } n < m\}.$$

Then there is a close connection between the number  $n$  being less than a number  $m$  and the corresponding pair  $\langle n, m \rangle$  being a member of  $R$ , namely,  $n < m$  if and only if  $\langle n, m \rangle \in R$ . In a sense we can consider the set  $R$  to be the  $<$ -relation on the set  $\mathbb{N}$ . In the same way we can construct a subset of  $\mathbb{N}^2$  for any relation between numbers. Conversely, given any set of pairs of numbers  $S \subseteq \mathbb{N}^2$ , there is a corresponding relation between numbers, namely, the relationship  $n$  bears to  $m$  if and only if  $\langle n, m \rangle \in S$ . This justifies the following definition:

**Definition rel.1** (Binary relation). A *binary relation* on a set  $X$  is a subset of  $X^2$ . If  $R \subseteq X^2$  is a binary relation on  $X$  and  $x, y \in X$ , we write  $Rxy$  (or  $xRy$ ) for  $\langle x, y \rangle \in R$ .

**Example rel.2.** The set  $\mathbb{N}^2$  of pairs of natural numbers can be listed in a 2-dimensional matrix like this: sfr:rel:set:  
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$$\begin{array}{ccccccc} \langle \mathbf{0}, \mathbf{0} \rangle & \langle 0, 1 \rangle & \langle 0, 2 \rangle & \langle 0, 3 \rangle & \dots & & \\ \langle 1, 0 \rangle & \langle \mathbf{1}, \mathbf{1} \rangle & \langle 1, 2 \rangle & \langle 1, 3 \rangle & \dots & & \\ \langle 2, 0 \rangle & \langle 2, 1 \rangle & \langle \mathbf{2}, \mathbf{2} \rangle & \langle 2, 3 \rangle & \dots & & \\ \langle 3, 0 \rangle & \langle 3, 1 \rangle & \langle 3, 2 \rangle & \langle \mathbf{3}, \mathbf{3} \rangle & \dots & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \end{array}$$

The subset consisting of the pairs lying on the diagonal, i.e.,

$$\{\langle 0, 0 \rangle, \langle 1, 1 \rangle, \langle 2, 2 \rangle, \dots\},$$

is the *identity relation* on  $\mathbb{N}$ . (Since the identity relation is popular, let's define  $\text{Id}_X = \{\langle x, x \rangle : x \in X\}$  for any set  $X$ .) The subset of all pairs lying above the diagonal, i.e.,

$$L = \{\langle 0, 1 \rangle, \langle 0, 2 \rangle, \dots, \langle 1, 2 \rangle, \langle 1, 3 \rangle, \dots, \langle 2, 3 \rangle, \langle 2, 4 \rangle, \dots\},$$

is the *less than* relation, i.e.,  $Lnm$  iff  $n < m$ . The subset of pairs below the diagonal, i.e.,

$$G = \{\langle 1, 0 \rangle, \langle 2, 0 \rangle, \langle 2, 1 \rangle, \langle 3, 0 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle, \dots\},$$

is the *greater than* relation, i.e.,  $Gnm$  iff  $n > m$ . The union of  $L$  with  $I$ ,  $K = L \cup I$ , is the *less than or equal to* relation:  $Knm$  iff  $n \leq m$ . Similarly,  $H = G \cup I$  is the *greater than or equal to* relation.  $L$ ,  $G$ ,  $K$ , and  $H$  are special kinds of relations called *orders*.  $L$  and  $G$  have the property that no number bears  $L$  or  $G$  to itself (i.e., for all  $n$ , neither  $Lnn$  nor  $Gnn$ ). Relations with this property are called *irreflexive*, and, if they also happen to be orders, they are called *strict orders*.

explanation Although orders and identity are important and natural relations, it should be emphasized that according to our definition *any* subset of  $X^2$  is a relation on  $X$ , regardless of how unnatural or contrived it seems. In particular,  $\emptyset$  is a relation on any set (the *empty relation*, which no pair of elements bears), and  $X^2$  itself is a relation on  $X$  as well (one which every pair bears), called the *universal relation*. But also something like  $E = \{\langle n, m \rangle : n > 5 \text{ or } m \times n \geq 34\}$  counts as a relation.

**Problem rel.1.** List the elements of the relation  $\subseteq$  on the set  $\wp(\{a, b, c\})$ .

## rel.2 Special Properties of Relations

intro Some kinds of relations turn out to be so common that they have been given special names. For instance,  $\leq$  and  $\subseteq$  both relate their respective domains (say,  $\mathbb{N}$  in the case of  $\leq$  and  $\wp(X)$  in the case of  $\subseteq$ ) in similar ways. To get at exactly how these relations are similar, and how they differ, we categorize them sfr:rel:prp:  
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according to some special properties that relations can have. It turns out that (combinations of) some of these special properties are especially important: orders and equivalence relations.

**Definition rel.3** (Reflexivity). A relation  $R \subseteq X^2$  is *reflexive* iff, for every  $x \in X$ ,  $Rxx$ .

**Definition rel.4** (Transitivity). A relation  $R \subseteq X^2$  is *transitive* iff, whenever  $Rxy$  and  $Ryz$ , then also  $Rxz$ .

**Definition rel.5** (Symmetry). A relation  $R \subseteq X^2$  is *symmetric* iff, whenever  $Rxy$ , then also  $Ryx$ .

**Definition rel.6** (Anti-symmetry). A relation  $R \subseteq X^2$  is *anti-symmetric* iff, whenever both  $Rxy$  and  $Ryx$ , then  $x = y$  (or, in other words: if  $x \neq y$  then either  $\neg Rxy$  or  $\neg Ryx$ ).

In a symmetric relation,  $Rxy$  and  $Ryx$  always hold together, or neither holds. In an anti-symmetric relation, the only way for  $Rxy$  and  $Ryx$  to hold together is if  $x = y$ . Note that this does not *require* that  $Rxy$  and  $Ryx$  holds when  $x = y$ , only that it isn't ruled out. So an anti-symmetric relation can be reflexive, but it is not the case that every anti-symmetric relation is reflexive. Also note that being anti-symmetric and merely not being symmetric are different conditions. In fact, a relation can be both symmetric and anti-symmetric at the same time (e.g., the identity relation is). explanation

**Definition rel.7** (Connectivity). A relation  $R \subseteq X^2$  is *connected* if for all  $x, y \in X$ , if  $x \neq y$ , then either  $Rxy$  or  $Ryx$ .

**Definition rel.8** (Partial order). A relation  $R \subseteq X^2$  that is reflexive, transitive, and anti-symmetric is called a *partial order*.

**Definition rel.9** (Linear order). A partial order that is also connected is called a *linear order*.

**Definition rel.10** (Equivalence relation). A relation  $R \subseteq X^2$  that is reflexive, symmetric, and transitive is called an *equivalence relation*.

**Problem rel.2.** Give examples of relations that are (a) reflexive and symmetric but not transitive, (b) reflexive and anti-symmetric, (c) anti-symmetric, transitive, but not reflexive, and (d) reflexive, symmetric, and transitive. Do not use relations on numbers or sets.

### rel.3 Orders

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sec Very often we are interested in comparisons between objects, where one object explanation may be less or equal or greater than another in a certain respect. Size is the most obvious example of such a comparative relation, or *order*. But not all such relations are alike in all their properties. For instance, some comparative

relations require any two objects to be comparable, others don't. (If they do, we call them *linear* or *total*.) Some include identity (like  $\leq$ ) and some exclude it (like  $<$ ). Let's get some order into all this.

**Definition rel.11** (Preorder). A relation which is both reflexive and transitive is called a *preorder*.

**Definition rel.12** (Partial order). A preorder which is also anti-symmetric is called a *partial order*.

**Definition rel.13** (Linear order). A partial order which is also connected is called a *total order* or *linear order*.

**Example rel.14.** Every linear order is also a partial order, and every partial order is also a preorder, but the converses don't hold. The universal relation on  $X$  is a preorder, since it is reflexive and transitive. But, if  $X$  has more than one **element**, the universal relation is not anti-symmetric, and so not a partial order. For a somewhat less silly example, consider the *no longer than* relation  $\preceq$  on  $\mathbb{B}^*$ :  $x \preceq y$  iff  $\text{len}(x) \leq \text{len}(y)$ . This is a preorder (reflexive and transitive), and even connected, but not a partial order, since it is not anti-symmetric. For instance,  $01 \preceq 10$  and  $10 \preceq 01$ , but  $01 \neq 10$ .

The relation of *divisibility without remainder* gives us an example of a partial order which isn't a linear order: for integers  $n, m$ , we say  $n$  (evenly) divides  $m$ , in symbols:  $n \mid m$ , if there is some  $k$  so that  $m = kn$ . On  $\mathbb{N}$ , this is a partial order, but not a linear order: for instance,  $2 \nmid 3$  and also  $3 \nmid 2$ . Considered as a relation on  $\mathbb{Z}$ , divisibility is only a preorder since anti-symmetry fails:  $1 \mid -1$  and  $-1 \mid 1$  but  $1 \neq -1$ . Another important partial order is the relation  $\subseteq$  on a set of sets.

Notice that the examples  $L$  and  $G$  from [Example rel.2](#), although we said there that they were called "strict orders," are not linear orders even though they are connected (they are not reflexive). But there is a close connection, as we will see momentarily.

**Definition rel.15** (Irreflexivity). A relation  $R$  on  $X$  is called *irreflexive* if, for all  $x \in X$ ,  $\neg Rxx$ .

**Definition rel.16** (Asymmetry). A relation  $R$  on  $X$  is called *asymmetric* if for no pair  $x, y \in X$  we have  $Rxy$  and  $Ryx$ .

**Definition rel.17** (Strict order). A *strict order* is a relation which is irreflexive, asymmetric, and transitive.

**Definition rel.18** (Strict linear order). A strict order which is also connected is called a *strict linear order*.

A strict order on  $X$  can be turned into a partial order by adding the diagonal  $\text{Id}_X$ , i.e., adding all the pairs  $\langle x, x \rangle$ . (This is called the *reflexive closure* of  $R$ .) Conversely, starting from a partial order, one can get a strict order by removing  $\text{Id}_X$ .

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### Proposition rel.19.

1. If  $R$  is a strict (linear) order on  $X$ , then  $R^+ = R \cup \text{Id}_X$  is a partial order (linear order).
2. If  $R$  is a partial order (linear order) on  $X$ , then  $R^- = R \setminus \text{Id}_X$  is a strict (linear) order.

*Proof.* 1. Suppose  $R$  is a strict order, i.e.,  $R \subseteq X^2$  and  $R$  is irreflexive, asymmetric, and transitive. Let  $R^+ = R \cup \text{Id}_X$ . We have to show that  $R^+$  is reflexive, antisymmetric, and transitive.

$R^+$  is clearly reflexive, since for all  $x \in X$ ,  $\langle x, x \rangle \in \text{Id}_X \subseteq R^+$ .

To show  $R^+$  is antisymmetric, suppose  $R^+xy$  and  $R^+yx$ , i.e.,  $\langle x, y \rangle$  and  $\langle y, x \rangle \in R^+$ , and  $x \neq y$ . Since  $\langle x, y \rangle \in R \cup \text{Id}_X$ , but  $\langle x, y \rangle \notin \text{Id}_X$ , we must have  $\langle x, y \rangle \in R$ , i.e.,  $Rxy$ . Similarly we get that  $Ryx$ . But this contradicts the assumption that  $R$  is asymmetric.

Now suppose that  $R^+xy$  and  $R^+yz$ . If both  $\langle x, y \rangle \in R$  and  $\langle y, z \rangle \in R$ , it follows that  $\langle x, z \rangle \in R$  since  $R$  is transitive. Otherwise, either  $\langle x, y \rangle \in \text{Id}_X$ , i.e.,  $x = y$ , or  $\langle y, z \rangle \in \text{Id}_X$ , i.e.,  $y = z$ . In the first case, we have that  $R^+yz$  by assumption,  $x = y$ , hence  $R^+xz$ . Similarly in the second case. In either case,  $R^+xz$ , thus,  $R^+$  is also transitive.

If  $R$  is connected, then for all  $x \neq y$ , either  $Rxy$  or  $Ryx$ , i.e., either  $\langle x, y \rangle \in R$  or  $\langle y, x \rangle \in R$ . Since  $R \subseteq R^+$ , this remains true of  $R^+$ , so  $R^+$  is connected as well.

2. Exercise. □

**Problem rel.3.** Complete the proof of [Proposition rel.19](#), i.e., prove that if  $R$  is a partial order on  $X$ , then  $R^- = R \setminus \text{Id}_X$  is a strict order.

**Example rel.20.**  $\leq$  is the linear order corresponding to the strict linear order  $<$ .  $\subseteq$  is the partial order corresponding to the strict order  $\subsetneq$ .

## rel.4 Graphs

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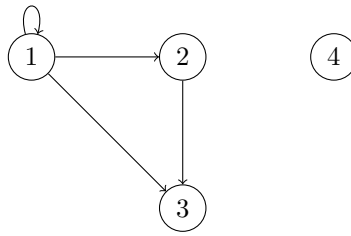
A *graph* is a diagram in which points—called “nodes” or “vertices” (plural of “vertex”)—are connected by edges. Graphs are a ubiquitous tool in discrete mathematics and in computer science. They are incredibly useful for representing, and visualizing, relationships and structures, from concrete things like networks of various kinds to abstract structures such as the possible outcomes of decisions. There are many different kinds of graphs in the literature which differ, e.g., according to whether the edges are directed or not, have labels or not, whether there can be edges from a node to the same node, multiple edges between the same nodes, etc. *Directed graphs* have a special connection to relations.

**Definition rel.21** (Directed graph). A *directed graph*  $G = \langle V, E \rangle$  is a set of *vertices*  $V$  and a set of *edges*  $E \subseteq V^2$ .

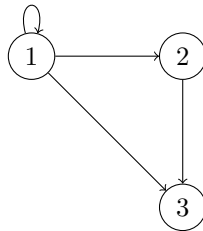
explanation

According to our definition, a graph just is a set together with a relation on that set. Of course, when talking about graphs, it's only natural to expect that they are graphically represented: we can draw a graph by connecting two vertices  $v_1$  and  $v_2$  by an arrow iff  $\langle v_1, v_2 \rangle \in E$ . The only difference between a relation by itself and a graph is that a graph specifies the set of vertices, i.e., a graph may have isolated vertices. The important point, however, is that every relation  $R$  on a set  $X$  can be seen as a directed graph  $\langle X, R \rangle$ , and conversely, a directed graph  $\langle V, E \rangle$  can be seen as a relation  $E \subseteq V^2$  with the set  $V$  explicitly specified.

**Example rel.22.** The graph  $\langle V, E \rangle$  with  $V = \{1, 2, 3, 4\}$  and  $E = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle\}$  looks like this:



This is a different graph than  $\langle V', E \rangle$  with  $V' = \{1, 2, 3\}$ , which looks like this:



**Problem rel.4.** Consider the less-than-or-equal-to relation  $\leq$  on the set  $\{1, 2, 3, 4\}$  as a graph and draw the corresponding diagram.

## rel.5 Operations on Relations

It is often useful to modify or combine relations. We've already used the union of relations above (which is just the union of two relations considered as sets of pairs). Here are some other ways: sfr:rel:ops:  
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**Definition rel.23.** Let  $R, S \subseteq X^2$  be relations and  $Y$  a set.

1. The *inverse*  $R^{-1}$  of  $R$  is  $R^{-1} = \{\langle y, x \rangle : \langle x, y \rangle \in R\}$ .

2. The *relative product*  $R \mid S$  of  $R$  and  $S$  is

$$(R \mid S) = \{\langle x, z \rangle : \text{for some } y, Rxy \text{ and } Syz\}$$

3. The *restriction*  $R \upharpoonright Y$  of  $R$  to  $Y$  is  $R \cap Y^2$

4. The *application*  $R[Y]$  of  $R$  to  $Y$  is

$$R[Y] = \{y : \text{for some } x \in Y, Rxy\}$$

**Example rel.24.** Let  $S \subseteq \mathbb{Z}^2$  be the successor relation on  $\mathbb{Z}$ , i.e., the set of pairs  $\langle x, y \rangle$  where  $x + 1 = y$ , for  $x, y \in \mathbb{Z}$ .  $Sxy$  holds iff  $y$  is the successor of  $x$ .

1. The inverse  $S^{-1}$  of  $S$  is the predecessor relation, i.e.,  $S^{-1}xy$  iff  $x - 1 = y$ .
2. The relative product  $S \mid S$  is the relation  $x$  bears to  $y$  if  $x + 2 = y$ .
3. The restriction of  $S$  to  $\mathbb{N}$  is the successor relation on  $\mathbb{N}$ .
4. The application of  $S$  to a set, e.g.,  $S[\{1, 2, 3\}]$  is  $\{2, 3, 4\}$ .

**Definition rel.25** (Transitive closure). The *transitive closure*  $R^+$  of a relation  $R \subseteq X^2$  is  $R^+ = \bigcup_{i=1}^{\infty} R^i$  where  $R^1 = R$  and  $R^{i+1} = R^i \mid R$ .

The *reflexive transitive closure* of  $R$  is  $R^* = R^+ \cup \text{Id}_X$ .

**Example rel.26.** Take the successor relation  $S \subseteq \mathbb{Z}^2$ .  $S^2xy$  iff  $x + 2 = y$ ,  $S^3xy$  iff  $x + 3 = y$ , etc. So  $R^*xy$  iff for some  $i \geq 1$ ,  $x + i = y$ . In other words,  $S^+xy$  iff  $x < y$  (and  $R^*xy$  iff  $x \leq y$ ).

**Problem rel.5.** Show that the transitive closure of  $R$  is in fact transitive.

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# Bibliography