Chapter udf

Relations

rel.1 Relations as Sets

sfr:rel:set:sec In **??**, we mentioned some important sets: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$. You will no doubt remember some interesting relations between the elements of some of these sets. For instance, each of these sets has a completely standard *order relation* on it. There is also the relation *is identical with* that every object bears to itself and to no other thing. There are many more interesting relations that we'll encounter, and even more possible relations. Before we review them, though, we will start by pointing out that we can look at relations as a special sort of set. explanation

For this, recall two things from **??**. First, recall the notion of a *ordered pair*: given a and b, we can form $\langle a, b\rangle$. Importantly, the order of elements *does* matter here. So if $a \neq b$ then $\langle a, b\rangle \neq \langle b, a\rangle$. (Contrast this with unordered pairs, i.e., 2-element sets, where $\{a, b\} = \{b, a\}$.) Second, recall the notion of a *Cartesian product*: if A and B are sets, then we can form $A \times B$, the set of all pairs $\langle x, y\rangle$ with $x \in A$ and $y \in B$. In particular, $A^2 = A \times A$ is the set of all ordered pairs from A.

Now we will consider a particular relation on a set: the $<$-relation on the set $\mathbb{N}$ of natural numbers. Consider the set of all pairs of numbers $\langle n, m\rangle$ where $n < m$, i.e.,

$$R = \{\langle n, m\rangle : n, m \in \mathbb{N} \text{ and } n < m\}.$$

There is a close connection between n being less than m, and the pair $\langle n, m\rangle$ being a member of R, namely:

$$n < m \text{ iff } \langle n, m\rangle \in R.$$

Indeed, without any loss of information, we can consider the set R to *be* the $<$-relation on $\mathbb{N}$.

In the same way we can construct a subset of $\mathbb{N}^2$ for any relation between numbers. Conversely, given any set of pairs of numbers $S \subseteq \mathbb{N}^2$, there is a corresponding relation between numbers, namely, the relationship n bears to m if and only if $\langle n, m\rangle \in S$. This justifies the following definition:

Definition rel.1 (Binary relation). A *binary relation* on a set A is a subset of A^2. If $R \subseteq A^2$ is a binary relation on A and $x, y \in A$, we sometimes write Rxy (or xRy) for $\langle x, y\rangle \in R$.

Example rel.2. The set $\mathbb{N}^2$ of pairs of natural numbers can be listed in a 2-dimensional matrix like this: sfr:rel:set:relations

$$\begin{array}{ccccc}
\langle \mathbf{0}, \mathbf{0}\rangle & \langle 0, 1\rangle & \langle 0, 2\rangle & \langle 0, 3\rangle & \dots \\
\langle 1, 0\rangle & \langle \mathbf{1}, \mathbf{1}\rangle & \langle 1, 2\rangle & \langle 1, 3\rangle & \dots \\
\langle 2, 0\rangle & \langle 2, 1\rangle & \langle \mathbf{2}, \mathbf{2}\rangle & \langle 2, 3\rangle & \dots \\
\langle 3, 0\rangle & \langle 3, 1\rangle & \langle 3, 2\rangle & \langle \mathbf{3}, \mathbf{3}\rangle & \dots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

We have put the diagonal, here, in bold, since the subset of $\mathbb{N}^2$ consisting of the pairs lying on the diagonal, i.e.,

$$\{\langle 0, 0\rangle, \langle 1, 1\rangle, \langle 2, 2\rangle, \dots\},$$

is the *identity relation on* $\mathbb{N}$. (Since the identity relation is popular, let's define $\mathrm{Id}_A = \{\langle x, x\rangle : x \in X\}$ for any set A.) The subset of all pairs lying above the diagonal, i.e.,

$$L = \{\langle 0, 1\rangle, \langle 0, 2\rangle, \dots, \langle 1, 2\rangle, \langle 1, 3\rangle, \dots, \langle 2, 3\rangle, \langle 2, 4\rangle, \dots\},$$

is the *less than* relation, i.e., Lnm iff $n < m$. The subset of pairs below the diagonal, i.e.,

$$G = \{\langle 1, 0\rangle, \langle 2, 0\rangle, \langle 2, 1\rangle, \langle 3, 0\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle, \dots\},$$

is the *greater than* relation, i.e., Gnm iff $n > m$. The union of L with I, which we might call $K = L \cup I$, is the *less than or equal to* relation: Knm iff $n \leq m$. Similarly, $H = G \cup I$ is the *greater than or equal to relation*. These relations L, G, K, and H are special kinds of relations called *orders*. L and G have the property that no number bears L or G to itself (i.e., for all n, neither Lnn nor Gnn). Relations with this property are called *irreflexive*, and, if they also happen to be orders, they are called *strict orders.*

explanation Although orders and identity are important and natural relations, it should be emphasized that according to our definition *any* subset of A^2 is a relation on A, regardless of how unnatural or contrived it seems. In particular, $\emptyset$ is a relation on any set (the *empty relation*, which no pair of elements bears), and A^2 itself is a relation on A as well (one which every pair bears), called the *universal relation*. But also something like $E = \{\langle n, m\rangle : n > 5 \text{ or } m \times n \geq 34\}$ counts as a relation.

Problem rel.1. List the elements of the relation $\subseteq$ on the set $\wp(\{a, b, c\})$.

rel.2 Philosophical Reflections

sfr:rel:ref:sec In section rel.1, we defined relations as certain sets. We should pause and ask a quick philosophical question: what is such a definition *doing*? It is extremely doubtful that we should want to say that we have *discovered* some metaphysical identity facts; that, for example, the order relation on $\mathbb{N}$ *turned out* to be the set $R = \{\langle n, m\rangle : n, m \in \mathbb{N} \text{ and } n < m\}$ we defined in section rel.1. Here are three reasons why.

First: in **??**, we defined $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$. Consider instead the definition $\|a, b\| = \{\{b\}, \{a, b\}\} = \langle b, a\rangle$. When $a \neq b$, we have that $\langle a, b\rangle \neq \|a, b\|$. But we could equally have regarded $\|a, b\|$ as our definition of an ordered pair, rather than $\langle a, b\rangle$. Both definitions would have worked equally well. So now we have two equally good candidates to "be" the order relation on the natural numbers, namely:

$$R = \{\langle n, m\rangle : n, m \in \mathbb{N} \text{ and } n < m\}$$
$$S = \{\|n, m\| : n, m \in \mathbb{N} \text{ and } n < m\}.$$

Since $R \neq S$, by extensionality, it is clear that they cannot *both* be identical to the order relation on $\mathbb{N}$. But it would just be arbitrary, and hence a bit embarrassing, to claim that R rather than S (or vice versa) *is* the ordering relation, as a matter of fact. (This is a very simple instance of an argument against set-theoretic reductionism which Benacerraf made famous in 1965. We will revisit it several times.)

Second: if we think that *every* relation should be identified with a set, then the relation of set-membership itself, $\in$, should be a particular set. Indeed, it would have to be the set $\{\langle x, y\rangle : x \in y\}$. But does this set exist? Given Russell's Paradox, it is a non-trivial claim that such a set exists. In fact, it is possible to develop set theory in a rigorous way as an axiomatic theory. In this theory, it will be provable that there is no set of all sets. So, even if some relations can be treated as sets, the relation of set-membership will have to be a special case.

Third: when we "identify" relations with sets, we said that we would allow ourselves to write Rxy for $\langle x, y\rangle \in R$. This is fine, provided that the membership relation, "$\in$", is treated *as* a predicate. But if we think that "$\in$" stands for a certain kind of set, then the expression "$\langle x, y\rangle \in R$" just consists of three singular terms which stand for sets: "$\langle x, y\rangle$", "$\in$", and "R". And such a list of names is no more capable of expressing a proposition than the nonsense string: "the cup penholder the table". Again, even if some relations can be treated as sets, the relation of set-membership must be a special case. (This rolls together a simple version of Frege's concept *horse* paradox, and a famous objection that Wittgenstein once raised against Russell.)

So where does this leave us? Well, there is nothing *wrong* with our saying that the relations on the numbers are sets. We just have to understand the spirit in which that remark is made. We are not stating a metaphysical identity fact. We are simply noting that, in certain contexts, we can (and will) *treat* (certain) relations as certain sets.

rel.3 Special Properties of Relations

intro Some kinds of relations turn out to be so common that they have been given special names. For instance, $\leq$ and $\subseteq$ both relate their respective domains (say, $\mathbb{N}$ in the case of $\leq$ and $\wp(A)$ in the case of $\subseteq$) in similar ways. To get at exactly how these relations are similar, and how they differ, we categorize them according to some special properties that relations can have. It turns out that (combinations of) some of these special properties are especially important: orders and equivalence relations. sfr:rel:prp:sec

Definition rel.3 (Reflexivity). A relation $R \subseteq A^2$ is *reflexive* iff, for every $x \in A$, Rxx.

Definition rel.4 (Transitivity). A relation $R \subseteq A^2$ is *transitive* iff, whenever Rxy and Ryz, then also Rxz.

Definition rel.5 (Symmetry). A relation $R \subseteq A^2$ is *symmetric* iff, whenever Rxy, then also Ryx.

Definition rel.6 (Anti-symmetry). A relation $R \subseteq A^2$ is *anti-symmetric* iff, whenever both Rxy and Ryx, then $x = y$ (or, in other words: if $x \neq y$ then either $\neg Rxy$ or $\neg Ryx$).

explanation In a symmetric relation, Rxy and Ryx always hold together, or neither holds. In an anti-symmetric relation, the only way for Rxy and Ryx to hold together is if $x = y$. Note that this does not *require* that Rxy and Ryx holds when $x = y$, only that it isn't ruled out. So an anti-symmetric relation can be reflexive, but it is not the case that every anti-symmetric relation is reflexive. Also note that being anti-symmetric and merely not being symmetric are different conditions. In fact, a relation can be both symmetric and anti-symmetric at the same time (e.g., the identity relation is).

Definition rel.7 (Connectivity). A relation $R \subseteq A^2$ is *connected* if for all $x, y \in A$, if $x \neq y$, then either Rxy or Ryx.

Problem rel.2. Give examples of relations that are (a) reflexive and symmetric but not transitive, (b) reflexive and anti-symmetric, (c) anti-symmetric, transitive, but not reflexive, and (d) reflexive, symmetric, and transitive. Do not use relations on numbers or sets.

Definition rel.8 (Irreflexivity). A relation $R \subseteq A^2$ is called *irreflexive* if, for all $x \in A$, not Rxx.

Definition rel.9 (Asymmetry). A relation $R \subseteq A^2$ is called *asymmetric* if for no pair $x, y \in A$ we have both Rxy and Ryx.

Note that if $A \neq \emptyset$, then no irreflexive relation on A is reflexive and every asymmetric relation on A is also anti-symmetric. However, there are $R \subseteq A^2$ that are not reflexive and also not irreflexive, and there are anti-symmetric relations that are not asymmetric.

rel.4 Equivalence Relations

sfr:rel:eqv: sec The identity relation on a set is reflexive, symmetric, and transitive. Relations R that have all three of these properties are very common.

Definition rel.10 (Equivalence relation). A relation $R \subseteq A^2$ that is reflexive, symmetric, and transitive is called an *equivalence relation*. Elements x and y of A are said to be *R-equivalent* if Rxy.

Equivalence relations give rise to the notion of an *equivalence class*. An equivalence relation "chunks up" the domain into different partitions. Within each partition, all the objects are related to one another; and no objects from different partitions relate to one another. Sometimes, it's helpful just to talk about these partitions *directly*. To that end, we introduce a definition:

sfr:rel:eqv: def:equivalenceclass **Definition rel.11.** Let $R \subseteq A^2$ be an equivalence relation. For each $x \in A$, the *equivalence class* of x in A is the set $[x]_R = \{y \in A : Rxy\}$. The *quotient* of A under R is $A/_R = \{[x]_R : x \in A\}$, i.e., the set of these equivalence classes.

The next result vindicates the definition of an equivalence class, in proving that the equivalence classes are indeed the partitions of A:

Proposition rel.12. *If $R \subseteq A^2$ is an equivalence relation, then Rxy iff $[x]_R = [y]_R$.*

Proof. For the left-to-right direction, suppose Rxy, and let $z \in [x]_R$. By definition, then, Rxz. Since R is an equivalence relation, Ryz. (Spelling this out: as Rxy and R is symmetric we have Ryx, and as Rxz and R is transitive we have Ryz.) So $z \in [y]_R$. Generalising, $[x]_R \subseteq [y]_R$. But exactly similarly, $[y]_R \subseteq [x]_R$. So $[x]_R = [y]_R$, by extensionality.

For the right-to-left direction, suppose $[x]_R = [y]_R$. Since R is reflexive, Ryy, so $y \in [y]_R$. Thus also $y \in [x]_R$ by the assumption that $[x]_R = [y]_R$. So Rxy. □

Example rel.13. A nice example of equivalence relations comes from modular arithmetic. For any a, b, and $n \in \mathbb{N}$, say that $a \equiv_n b$ iff dividing a by n gives remainder b. (Somewhat more symbolically: $a \equiv_n b$ iff $(\exists k \in \mathbb{N}) a - b = kn$.) Now, $\equiv_n$ is an equivalence relation, for any n. And there are exactly n distinct equivalence classes generated by $\equiv_n$; that is, $\mathbb{N}/_{\equiv_n}$ has n elements. These are: the set of numbers divisible by n without remainder, i.e., $[0]_{\equiv_n}$; the set of numbers divisible by n with remainder 1, i.e., $[1]_{\equiv_n}$; …; and the set of numbers divisible by n with remainder $n-1$, i.e., $[n-1]_{\equiv_n}$.

Problem rel.3. Show that $\equiv_n$ is an equivalence relation, for any $n \in \mathbb{N}$, and that $\mathbb{N}/_{\equiv_n}$ has exactly n members.

rel.5 Orders

explanation Many of our comparisons involve describing some objects as being "less than", "equal to", or "greater than" other objects, in a certain respect. These involve *order* relations. But there are different kinds of order relations. For instance, some require that any two objects be comparable, others don't. Some include identity (like $\leq$) and some exclude it (like $<$). It will help us to have a taxonomy here. sfr:rel:ord:sec

Definition rel.14 (Preorder). A relation which is both reflexive and transitive is called a *preorder.*

Definition rel.15 (Partial order). A preorder which is also anti-symmetric is called a *partial order.*

Definition rel.16 (Linear order). A partial order which is also connected is called a *total order* or *linear order.*

Every linear order is also a partial order, and every partial order is also a preorder, but the converses don't hold.

Example rel.17. Every linear order is also a partial order, and every partial order is also a preorder, but the converses don't hold. The universal relation on A is a preorder, since it is reflexive and transitive. But, if A has more than one element, the universal relation is not anti-symmetric, and so not a partial order.

Example rel.18. Consider the *no longer than* relation $\preccurlyeq$ on $\mathbb{B}^*$: $x \preccurlyeq y$ iff $\text{len}(x) \leq \text{len}(y)$. This is a preorder (reflexive and transitive), and even connected, but not a partial order, since it is not anti-symmetric. For instance, $01 \preccurlyeq 10$ and $10 \preccurlyeq 01$, but $01 \neq 10$.

Example rel.19. An important partial order is the relation $\subseteq$ on a set of sets. This is not in general a linear order, since if $a \neq b$ and we consider $\wp(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, we see that $\{a\} \nsubseteq \{b\}$ and $\{a\} \neq \{b\}$ and $\{b\} \nsubseteq \{a\}$.

Example rel.20. The relation of *divisibility without remainder* gives us a partial order which isn't a linear order. For integers n, m, we write $n \mid m$ to mean n (evenly) divides m, i.e., iff there is some integer k so that $m = kn$. On $\mathbb{N}$, this is a partial order, but not a linear order: for instance, $2 \nmid 3$ and also $3 \nmid 2$. Considered as a relation on $\mathbb{Z}$, divisibility is only a preorder since it is not anti-symmetric: $1 \mid -1$ and $-1 \mid 1$ but $1 \neq -1$.

Definition rel.21 (Strict order). A *strict order* is a relation which is irreflexive, asymmetric, and transitive.

Definition rel.22 (Strict linear order). A strict order which is also connected is called a *strict linear order.*

Example rel.23. $\leq$ is the linear order corresponding to the strict linear order $<$. $\subseteq$ is the partial order corresponding to the strict order $\subsetneq$.

sfr:rel:ord: def:strictlinearorder

Definition rel.24 (Total order). A strict order which is also connected is called a *total order*. This is also sometimes called a *strict linear order*.

Any strict order R on A can be turned into a partial order by adding the diagonal Id_A, i.e., adding all the pairs $\langle x, x\rangle$. (This is called the *reflexive closure* of R.) Conversely, starting from a partial order, one can get a strict order by removing Id_A. These next two results make this precise.

sfr:rel:ord: prop:stricttopartial

Proposition rel.25. *If R is a strict order on A, then $R^+ = R \cup \text{Id}_A$ is a partial order. Moreover, if R is total, then R^+ is a linear order.*

Proof. Suppose R is a strict order, i.e., $R \subseteq A^2$ and R is irreflexive, asymmetric, and transitive. Let $R^+ = R \cup \text{Id}_A$. We have to show that R^+ is reflexive, antisymmetric, and transitive.

R^+ is clearly reflexive, since $\langle x, x\rangle \in \text{Id}_A \subseteq R^+$ for all $x \in A$.

To show R^+ is antisymmetric, suppose for reductio that R^+xy and R^+yx but $x \neq y$. Since $\langle x, y\rangle \in R \cup \text{Id}_X$, but $\langle x, y\rangle \notin \text{Id}_X$, we must have $\langle x, y\rangle \in R$, i.e., Rxy. Similarly, Ryx. But this contradicts the assumption that R is asymmetric.

To establish transitivity, suppose that R^+xy and R^+yz. If both $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$, then $\langle x, z\rangle \in R$ since R is transitive. Otherwise, either $\langle x, y\rangle \in \text{Id}_X$, i.e., $x = y$, or $\langle y, z\rangle \in \text{Id}_X$, i.e., $y = z$. In the first case, we have that R^+yz by assumption, $x = y$, hence R^+xz. Similarly in the second case. In either case, R^+xz, thus, R^+ is also transitive.

Concerning the "moreover" clause, suppose R is a total order, i.e., that R is connected. So for all $x \neq y$, either Rxy or Ryx, i.e., either $\langle x, y\rangle \in R$ or $\langle y, x\rangle \in R$. Since $R \subseteq R^+$, this remains true of R^+, so R^+ is connected as well. □

sfr:rel:ord: prop:partialtostrict

Proposition rel.26. *If R is a partial order on X, then $R^- = R \setminus \text{Id}_X$ is a strict order. Moreover, if R is linear, then R^- is total.*

Proof. This is left as an exercise. □

Problem rel.4. Give a proof of Proposition rel.26.

Example rel.27. $\leq$ is the linear order corresponding to the total order $<$. $\subseteq$ is the partial order corresponding to the strict order $\subsetneq$.

The following simple result which establishes that total orders satisfy an extensionality-like property:

sfr:rel:ord: prop:extensionality-totalorders

Proposition rel.28. *If $<$ totally orders A, then:*

$$(\forall a, b \in A)((\forall x \in A)(x < a \leftrightarrow x < b) \rightarrow a = b)$$

Proof. Suppose $(\forall x \in A)(x < a \leftrightarrow x < b)$. If $a < b$, then $a < a$, contradicting the fact that $<$ is irreflexive; so $a \not< b$. Exactly similarly, $b \not< a$. So $a = b$, as $<$ is connected. □

rel.6 Graphs

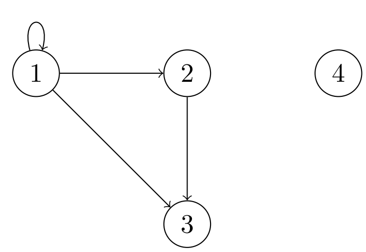
A *graph* is a diagram in which points—called "nodes" or "vertices" (plural of "vertex")—are connected by edges. Graphs are a ubiquitous tool in discrete mathematics and in computer science. They are incredibly useful for representing, and visualizing, relationships and structures, from concrete things like networks of various kinds to abstract structures such as the possible outcomes of decisions. There are many different kinds of graphs in the literature which differ, e.g., according to whether the edges are directed or not, have labels or not, whether there can be edges from a node to the same node, multiple edges between the same nodes, etc. *Directed graphs* have a special connection to relations. sfr:rel:grp:sec

Definition rel.29 (Directed graph). A *directed graph* $G = \langle V, E\rangle$ is a set of *vertices* V and a set of *edges* $E \subseteq V^2$.

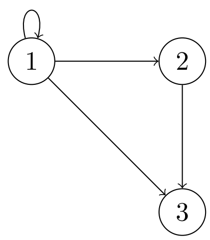
explanation

According to our definition, a graph just is a set together with a relation on that set. Of course, when talking about graphs, it's only natural to expect that they are graphically represented: we can draw a graph by connecting two vertices v_1 and v_2 by an arrow iff $\langle v_1, v_2\rangle \in E$. The only difference between a relation by itself and a graph is that a graph specifies the set of vertices, i.e., a graph may have isolated vertices. The important point, however, is that every relation R on a set X can be seen as a directed graph $\langle X, R\rangle$, and conversely, a directed graph $\langle V, E\rangle$ can be seen as a relation $E \subseteq V^2$ with the set V explicitly specified.

Example rel.30. The graph $\langle V, E\rangle$ with $V = \{1, 2, 3, 4\}$ and $E = \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle\}$ looks like this:



This is a different graph than $\langle V', E\rangle$ with $V' = \{1, 2, 3\}$, which looks like this:



Problem rel.5. Consider the less-than-or-equal-to relation $\leq$ on the set $\{1, 2, 3, 4\}$ as a graph and draw the corresponding diagram.

rel.7 Operations on Relations

sfr:rel:ops:sec It is often useful to modify or combine relations. In Proposition rel.25, we considered the *union* of relations, which is just the union of two relations considered as sets of pairs. Similarly, in Proposition rel.26, we considered the relative difference of relations. Here are some other operations we can perform on relations.

sfr:rel:ops:relationoperations **Definition rel.31.** Let R, S be relations, and A be any set.

The *inverse* of R is $R^{-1} = \{\langle y, x\rangle : \langle x, y\rangle \in R\}$.

The *relative product* of R and S is $(R \mid S) = \{\langle x, z\rangle : \exists y(Rxy \land Syz)\}$.

The *restriction* of R to A is $R\restriction_A = R \cap A^2$.

The *application* of R to A is $R[A] = \{y : (\exists x \in A)Rxy\}$

Example rel.32. Let $S \subseteq \mathbb{Z}^2$ be the successor relation on $\mathbb{Z}$, i.e., $S = \{\langle x, y\rangle \in \mathbb{Z}^2 : x + 1 = y\}$, so that Sxy iff $x + 1 = y$.

S^{-1} is the predecessor relation on $\mathbb{Z}$, i.e., $\{\langle x, y\rangle \in \mathbb{Z}^2 : x - 1 = y\}$.

$S \mid S$ is $\{\langle x, y\rangle \in \mathbb{Z}^2 : x + 2 = y\}$

$S\restriction_{\mathbb{N}}$ is the successor relation on $\mathbb{N}$.

$S[\{1, 2, 3\}]$ is $\{2, 3, 4\}$.

Definition rel.33 (Transitive closure). Let $R \subseteq A^2$ be a binary relation.

The *transitive closure* of R is $R^+ = \bigcup_{0<n\in\mathbb{N}} R^n$, where we recursively define $R^1 = R$ and $R^{n+1} = R^n \mid R$.

The *reflexive transitive closure* of R is $R^* = R^+ \cup \mathrm{Id}_X$.

Example rel.34. Take the successor relation $S \subseteq \mathbb{Z}^2$. S^2xy iff $x + 2 = y$, S^3xy iff $x + 3 = y$, etc. So S^+xy iff $x + n = y$ for some $n > 1$. In other words, S^+xy iff $x < y$, and S^*xy iff $x \leq y$.

Problem rel.6. Show that the transitive closure of R is in fact transitive.

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Bibliography

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