rel.1 Philosophical Reflections

sfr:rel:ref:sec In **??**, we defined relations as certain sets. We should pause and ask a quick philosophical question: what is such a definition *doing*? It is extremely doubtful that we should want to say that we have *discovered* some metaphysical identity facts; that, for example, the order relation on $\mathbb{N}$ *turned out* to be the set $R = \{\langle n, m\rangle : n, m \in \mathbb{N} \text{ and } n < m\}$ we defined in **??**. Here are three reasons why.

First: in **??**, we defined $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$. Consider instead the definition $\|a, b\| = \{\{b\}, \{a, b\}\} = \langle b, a\rangle$. When $a \neq b$, we have that $\langle a, b\rangle \neq \|a, b\|$. But we could equally have regarded $\|a, b\|$ as our definition of an ordered pair, rather than $\langle a, b\rangle$. Both definitions would have worked equally well. So now we have two equally good candidates to "be" the order relation on the natural numbers, namely:

$$R = \{\langle n, m\rangle : n, m \in \mathbb{N} \text{ and } n < m\}$$
$$S = \{\|n, m\| : n, m \in \mathbb{N} \text{ and } n < m\}.$$

Since $R \neq S$, by extensionality, it is clear that they cannot *both* be identical to the order relation on $\mathbb{N}$. But it would just be arbitrary, and hence a bit embarrassing, to claim that R rather than S (or vice versa) *is* the ordering relation, as a matter of fact. (This is a very simple instance of an argument against set-theoretic reductionism which Benacerraf made famous in 1965. We will revisit it several times.)

Second: if we think that *every* relation should be identified with a set, then the relation of set-membership itself, $\in$, should be a particular set. Indeed, it would have to be the set $\{\langle x, y\rangle : x \in y\}$. But does this set exist? Given Russell's Paradox, it is a non-trivial claim that such a set exists. In fact, it is possible to develop set theory in a rigorous way as an axiomatic theory. In this theory, it will be provable that there is no set of all sets. So, even if some relations can be treated as sets, the relation of set-membership will have to be a special case.

Third: when we "identify" relations with sets, we said that we would allow ourselves to write Rxy for $\langle x, y\rangle \in R$. This is fine, provided that the membership relation, "$\in$", is treated *as* a predicate. But if we think that "$\in$" stands for a certain kind of set, then the expression "$\langle x, y\rangle \in R$" just consists of three singular terms which stand for sets: "$\langle x, y\rangle$", "$\in$", and "R". And such a list of names is no more capable of expressing a proposition than the nonsense string: "the cup penholder the table". Again, even if some relations can be treated as sets, the relation of set-membership must be a special case. (This rolls together a simple version of Frege's concept *horse* paradox, and a famous objection that Wittgenstein once raised against Russell.)

So where does this leave us? Well, there is nothing *wrong* with our saying that the relations on the numbers are sets. We just have to understand the spirit in which that remark is made. We are not stating a metaphysical identity fact. We are simply noting that, in certain contexts, we can (and will) *treat* (certain) relations as certain sets.

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Bibliography

Benacerraf, Paul. 1965. What numbers could not be. *The Philosophical Review* 74(1): 47–73.