Many of our comparisons involve describing some objects as being “less than”, “equal to”, or “greater than” other objects, in a certain respect. These involve order relations. But there are different kinds of order relations. For instance, some require that any two objects be comparable, others don’t. Some include identity (like ≤) and some exclude it (like <). It will help us to have a taxonomy here.

**Definition rel.1 (Preorder).** A relation which is both reflexive and transitive is called a preorder.

**Definition rel.2 (Partial order).** A preorder which is also anti-symmetric is called a partial order.

**Definition rel.3 (Linear order).** A partial order which is also connected is called a total order or linear order.

Every linear order is also a partial order, and every partial order is also a preorder, but the converses don’t hold.

**Example rel.4.** Every linear order is also a partial order, and every partial order is also a preorder, but the converses don’t hold. The universal relation on \( A \) is a preorder, since it is reflexive and transitive. But, if \( A \) has more than one element, the universal relation is not anti-symmetric, and so not a partial order.

**Example rel.5.** Consider the no longer than relation \( \preceq \) on \( \mathbb{B}^* \): \( x \preceq y \) iff \( \text{len}(x) \leq \text{len}(y) \). This is a preorder (reflexive and transitive), and even connected, but not a partial order, since it is not anti-symmetric. For instance, \( 01 \preceq 10 \) and \( 10 \preceq 01 \), but \( 01 \neq 10 \).

**Example rel.6.** An important partial order is the relation \( \subseteq \) on a set of sets. This is not in general a linear order, since if \( a \neq b \) and we consider \( \varphi(\{a, b\}) = \emptyset, \{a\}, \{b\}, \{a, b\} \), we see that \( \{a\} \not\subseteq \{b\} \) and \( \{a\} \neq \{b\} \) and \( \{b\} \not\subseteq \{a\} \).

**Example rel.7.** The relation of divisibility without remainder gives us a partial order which isn’t a linear order. For integers \( n, m \), we write \( n \mid m \) to mean \( n \) (evenly) divides \( m \), i.e., iff there is some integer \( k \) so that \( m = kn \). On \( \mathbb{N} \), this is a partial order, but not a linear order: for instance, \( 2 \nmid 3 \) and also \( 3 \nmid 2 \). Considered as a relation on \( \mathbb{Z} \), divisibility is only a preorder since it is not anti-symmetric: \( 1 \mid -1 \) and \( -1 \mid 1 \) but \( 1 \neq -1 \).

**Definition rel.8 (Strict order).** A strict order is a relation which is irreflexive, asymmetric, and transitive.

**Definition rel.9 (Strict linear order).** A strict order which is also connected is called a strict linear order.
Example rel.10. \( \leq \) is the linear order corresponding to the strict linear order \(<\). \( \subseteq \) is the partial order corresponding to the strict order \(\subset\).

Definition rel.11 (Total order). A strict order which is also connected is called a total order. This is also sometimes called a strict linear order.

Any strict order \( R \) on \( A \) can be turned into a partial order by adding the diagonal \( \text{Id}_A \), i.e., adding all the pairs \( (x, x) \). (This is called the reflexive closure of \( R \).) Conversely, starting from a partial order, one can get a strict order by removing \( \text{Id}_A \). These next two results make this precise.

Proposition rel.12. If \( R \) is a strict order on \( A \), then \( R^+ = R \cup \text{Id}_A \) is a partial order. Moreover, if \( R \) is total, then \( R^+ \) is a linear order.

Proof. Suppose \( R \) is a strict order, i.e., \( R \subseteq A^2 \) and \( R \) is irreflexive, asymmetric, and transitive. Let \( R^+ = R \cup \text{Id}_A \). We have to show that \( R^+ \) is reflexive, antisymmetric, and transitive.

\( R^+ \) is clearly reflexive, since \( (x, x) \in \text{Id}_A \subseteq R^+ \) for all \( x \in A \).

To show \( R^+ \) is antisymmetric, suppose for reductio that \( R^+ \) itself is a strict order, i.e., \( R \cup \text{Id}_A \), but \( x \neq y \). Since \( (x, y) \in R \cup \text{Id}_X \), but \( (x, y) \notin \text{Id}_X \), we must have \( (x, y) \in R \), i.e., \( Rxy \). Similarly, \( Ryx \). But this contradicts the assumption that \( R \) is asymmetric.

To establish transitivity, suppose that \( R^+ xy \) and \( R^+ yz \). If both \( (x, y) \in R \) and \( (y, z) \in R \), then \( (x, z) \in R \) since \( R \) is transitive. Otherwise, either \( (x, y) \in \text{Id}_X \), i.e., \( x = y \), or \( (y, z) \in \text{Id}_X \), i.e., \( y = z \). In the first case, we have that \( R^+ yz \) by assumption, \( x = y \), hence \( R^+ xz \). Similarly in the second case. In either case, \( R^+ xz \), thus, \( R^+ \) is also transitive.

Concerning the “moreover” clause, suppose \( R \) is a total order, i.e., that \( R \) is connected. So for all \( x \neq y \), either \( Rxy \) or \( Ryx \), i.e., either \( (x, y) \in R \) or \( (y, x) \in R \). Since \( R \subseteq R^+ \), this remains true of \( R^+ \), so \( R^+ \) is connected as well. \( \square \)

Proposition rel.13. If \( R \) is a partial order on \( X \), then \( R^- = R \setminus \text{Id}_X \) is a strict order. Moreover, if \( R \) is linear, then \( R^- \) is total.

Proof. This is left as an exercise. \( \square \)

Problem rel.1. Give a proof of Proposition rel.13.

Example rel.14. \( \leq \) is the linear order corresponding to the total order \(<\). \( \subseteq \) is the partial order corresponding to the strict order \(\subset\).

The following simple result which establishes that total orders satisfy an extensionality-like property:

Proposition rel.15. If \( < \) totally orders \( A \), then:

\[(\forall a,b \in A)((\forall x \in A)(x < a \leftrightarrow x < b) \rightarrow a = b)\]
Proof. Suppose $(\forall x \in A)(x < a \leftrightarrow x < b)$. If $a < b$, then $a < a$, contradicting the fact that $<$ is irreflexive; so $a \not< b$. Exactly similarly, $b \not< a$. So $a = b$, as $<$ is connected.

\[ \square \]

Photo Credits

Bibliography