rel.1 Operations on Relations

sfr:rel:ops: It is often useful to modify or combine relations. In ??, we considered the *union* of relations, which is just the union of two relations considered as sets of pairs. Similarly, in ??, we considered the relative difference of relations. Here are some other operations we can perform on relations.

sfr:rel:ops: **Definition rel.1.** Let R, S be relations, and A be any set.

The inverse of R is $R^{-1} = \{\langle y, x \rangle : \langle x, y \rangle \in R\}.$

The relative product of R and S is $(R \mid S) = \{\langle x, z \rangle : \exists y(Rxy \land Syz)\}$. The restriction of R to A is $R \upharpoonright_A = R \cap A^2$. The application of R to A is $R[A] = \{y : (\exists x \in A)Rxy\}$

Example rel.2. Let $S \subseteq \mathbb{Z}^2$ be the successor relation on \mathbb{Z} , i.e., $S = \{\langle x, y \rangle \in \mathbb{Z}^2 : x + 1 = y\}$, so that Sxy iff x + 1 = y. S^{-1} is the predecessor relation on \mathbb{Z} , i.e., $\{\langle x, y \rangle \in \mathbb{Z}^2 : x - 1 = y\}$. $S \mid S$ is $\{\langle x, y \rangle \in \mathbb{Z}^2 : x + 2 = y\}$ $S \upharpoonright_{\mathbb{N}}$ is the successor relation on \mathbb{N} .

 $S[\{1, 2, 3\}]$ is $\{2, 3, 4\}$.

relationoperations

Definition rel.3 (Transitive closure). Let $R \subseteq A^2$ be a binary relation. The transitive closure of R is $R^+ = \bigcup_{0 \le n \in \mathbb{N}} R^n$, where we recursively define $R^1 = R$ and $R^{n+1} = R^n \mid R$. The reflexive transitive closure of R is $R^* = R^+ \cup \mathrm{Id}_A$.

Example rel.4. Take the successor relation $S \subseteq \mathbb{Z}^2$. S^2xy iff x + 2 = y, S^3xy iff x + 3 = y, etc. So S^+xy iff x + n = y for some $n \ge 1$. In other words, S^+xy iff x < y, and S^*xy iff $x \le y$.

Problem rel.1. Show that the transitive closure of R is in fact transitive.

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Bibliography