

## rel.1 Operations on Relations

sfr:rel:ops:  
sec It is often useful to modify or combine relations. In ??, we considered the *union* of relations, which is just the union of two relations considered as sets of pairs. Similarly, in ??, we considered the relative difference of relations. Here are some other operations we can perform on relations.

sfr:rel:ops:  
relationoperations **Definition rel.1.** Let  $R, S$  be relations, and  $A$  be any set.

The *inverse* of  $R$  is  $R^{-1} = \{\langle y, x \rangle : \langle x, y \rangle \in R\}$ .

The *relative product* of  $R$  and  $S$  is  $(R \mid S) = \{\langle x, z \rangle : \exists y(Rxy \wedge Syz)\}$ .

The *restriction* of  $R$  to  $A$  is  $R \upharpoonright_A = R \cap A^2$ .

The *application* of  $R$  to  $A$  is  $R[A] = \{y : (\exists x \in A)Rxy\}$

**Example rel.2.** Let  $S \subseteq \mathbb{Z}^2$  be the successor relation on  $\mathbb{Z}$ , i.e.,  $S = \{\langle x, y \rangle \in \mathbb{Z}^2 : x + 1 = y\}$ , so that  $Sxy$  iff  $x + 1 = y$ .

$S^{-1}$  is the predecessor relation on  $\mathbb{Z}$ , i.e.,  $\{\langle x, y \rangle \in \mathbb{Z}^2 : x - 1 = y\}$ .

$S \mid S$  is  $\{\langle x, y \rangle \in \mathbb{Z}^2 : x + 2 = y\}$

$S \upharpoonright_{\mathbb{N}}$  is the successor relation on  $\mathbb{N}$ .

$S[\{1, 2, 3\}]$  is  $\{2, 3, 4\}$ .

**Definition rel.3 (Transitive closure).** Let  $R \subseteq A^2$  be a binary relation.

The *transitive closure* of  $R$  is  $R^+ = \bigcup_{0 < n \in \mathbb{N}} R^n$ , where we recursively define  $R^1 = R$  and  $R^{n+1} = R^n \mid R$ .

The *reflexive transitive closure* of  $R$  is  $R^* = R^+ \cup \text{Id}_X$ .

**Example rel.4.** Take the successor relation  $S \subseteq \mathbb{Z}^2$ .  $S^2xy$  iff  $x + 2 = y$ ,  $S^3xy$  iff  $x + 3 = y$ , etc. So  $S^+xy$  iff  $x + n = y$  for some  $n > 1$ . In other words,  $S^+xy$  iff  $x < y$ , and  $S^*xy$  iff  $x \leq y$ .

**Problem rel.1.** Show that the transitive closure of  $R$  is in fact transitive.

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## Bibliography