It is often useful to modify or combine relations. In ??, we considered the
union of relations, which is just the union of two relations considered as sets of
pairs. Similarly, in ??, we considered the relative difference of relations. Here
are some other operations we can perform on relations.

**Definition rel.1.** Let $R$, $S$ be relations, and $A$ be any set.
- The *inverse* of $R$ is $R^{-1} = \{ \langle y, x \rangle : \langle x, y \rangle \in R \}$.
- The *relative product* of $R$ and $S$ is $(R \mid S) = \{ \langle x, z \rangle : \exists y (Rxy \land Syz) \}$.
- The *restriction* of $R$ to $A$ is $R \upharpoonright A = R \cap A^2$.
- The *application* of $R$ to $A$ is $R[A] = \{ y : (\exists x \in A) Rxy \}$.

**Example rel.2.** Let $S \subseteq \mathbb{Z}^2$ be the successor relation on $\mathbb{Z}$, i.e., $S = \{ \langle x, y \rangle \in \mathbb{Z}^2 : x + 1 = y \}$, so that $Sxy$ iff $x + 1 = y$.
- $S^{-1}$ is the predecessor relation on $\mathbb{Z}$, i.e., $\{ \langle x, y \rangle \in \mathbb{Z}^2 : x - 1 = y \}$.
- $S \mid S$ is $\{ \langle x, y \rangle \in \mathbb{Z}^2 : x + 2 = y \}$
- $S \upharpoonright \mathbb{N}$ is the successor relation on $\mathbb{N}$.
- $S[\{1, 2, 3\}] = \{2, 3, 4\}$.

**Definition rel.3 (Transitive closure).** Let $R \subseteq A^2$ be a binary relation.
- The *transitive closure* of $R$ is $R^+ = \bigcup_{n \in \mathbb{N}} R^n$, where we recursively define $R^1 = R$ and $R^{n+1} = R^n \mid R$.
- The *reflexive transitive closure* of $R$ is $R^* = R^+ \cup \text{Id}_A$.

**Example rel.4.** Take the successor relation $S \subseteq \mathbb{Z}^2$. $S^2xy$ iff $x + 2 = y$, $S^3xy$ iff $x + 3 = y$, etc. So $S^+xy$ iff $x + n = y$ for some $n \geq 1$. In other words, $S^+xy$ iff $x < y$, and $S^*xy$ iff $x \leq y$.

**Problem rel.1.** Show that the transitive closure of $R$ is in fact transitive.

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**Bibliography**