rel.1 Equivalence Relations

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The identity relation on a set is reflexive, symmetric, and transitive. Relations R that have all three of these properties are very common.

Definition rel.1 (Equivalence relation). A relation $R \subseteq A^2$ that is reflexive, symmetric, and transitive is called an *equivalence relation*. Elements x and y of A are said to be *R-equivalent* if Rxy.

Equivalence relations give rise to the notion of an *equivalence class*. An equivalence relation "chunks up" the domain into different partitions. Within each partition, all the objects are related to one another; and no objects from different partitions relate to one another. Sometimes, it's helpful just to talk about these partitions *directly*. To that end, we introduce a definition:

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Definition rel.2. Let $R \subseteq A^2$ be an equivalence relation. For each $x \in A$, the *equivalence class* of x in A is the set $[x]_R = \{y \in A : Rxy\}$. The *quotient* of A under R is $A/_R = \{[x]_R : x \in A\}$, i.e., the set of these equivalence classes.

The next result vindicates the definition of an equivalence class, in proving that the equivalence classes are indeed the partitions of A:

Proposition rel.3. *If $R \subseteq A^2$ is an equivalence relation, then Rxy iff $[x]_R = [y]_R$.*

Proof. For the left-to-right direction, suppose Rxy, and let $z \in [x]_R$. By definition, then, Rxz. Since R is an equivalence relation, Ryz. (Spelling this out: as Rxy and R is symmetric we have Ryx, and as Rxz and R is transitive we have Ryz.) So $z \in [y]_R$. Generalising, $[x]_R \subseteq [y]_R$. But exactly similarly, $[y]_R \subseteq [x]_R$. So $[x]_R = [y]_R$, by extensionality.

For the right-to-left direction, suppose $[x]_R = [y]_R$. Since R is reflexive, Ryy, so $y \in [y]_R$. Thus also $y \in [x]_R$ by the assumption that $[x]_R = [y]_R$. So Rxy. □

Example rel.4. A nice example of equivalence relations comes from modular arithmetic. For any a, b, and $n \in \mathbb{N}$, say that $a \equiv_n b$ iff dividing a by n gives remainder b. (Somewhat more symbolically: $a \equiv_n b$ iff $(\exists k \in \mathbb{N}) a - b = kn$.) Now, $\equiv_n$ is an equivalence relation, for any n. And there are exactly n distinct equivalence classes generated by $\equiv_n$; that is, $\mathbb{N}/_{\equiv_n}$ has n elements. These are: the set of numbers divisible by n without remainder, i.e., $[0]_{\equiv_n}$; the set of numbers divisible by n with remainder 1, i.e., $[1]_{\equiv_n}$; ...; and the set of numbers divisible by n with remainder $n - 1$, i.e., $[n-1]_{\equiv_n}$.

Problem rel.1. Show that $\equiv_n$ is an equivalence relation, for any $n \in \mathbb{N}$, and that $\mathbb{N}/_{\equiv_n}$ has exactly n members.

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Bibliography