Equivalence Relations

The identity relation on a set is reflexive, symmetric, and transitive. Relations \( R \) that have all three of these properties are very common.

**Definition rel.1 (Equivalence relation).** A relation \( R \subseteq A^2 \) that is reflexive, symmetric, and transitive is called an *equivalence relation*. Elements \( x \) and \( y \) of \( A \) are said to be \( R \)-equivalent if \( Rxy \).

Equivalence relations give rise to the notion of an *equivalence class*. An equivalence relation “chunks up” the domain into different partitions. Within each partition, all the objects are related to one another; and no objects from different partitions relate to one another. Sometimes, it’s helpful just to talk about these partitions directly. To that end, we introduce a definition:

**Definition rel.2.** Let \( R \subseteq A^2 \) be an equivalence relation. For each \( x \in A \), the *equivalence class* of \( x \) in \( A \) is the set \( [x]_R = \{ y \in A : Rxy \} \). The *quotient* of \( A \) under \( R \) is \( A/R = \{ [x]_R : x \in A \} \), i.e., the set of these equivalence classes.

The next result vindicates the definition of an equivalence class, in proving that the equivalence classes are indeed the partitions of \( A \):

**Proposition rel.3.** If \( R \subseteq A^2 \) is an equivalence relation, then \( Rxy \) iff \( [x]_R = [y]_R \).

*Proof.* For the left-to-right direction, suppose \( Rxy \), and let \( z \in [x]_R \). By definition, then, \( Rxz \). Since \( R \) is an equivalence relation, \( Ryz \). (Spelling this out: as \( Rxy \) and \( R \) is symmetric we have \( Ryx \), and as \( Rxz \) and \( R \) is transitive we have \( Ryz \).) So \( z \in [y]_R \). Generalising, \( [x]_R \subseteq [y]_R \). But exactly similarly, \( [y]_R \subseteq [x]_R \). So \( [x]_R = [y]_R \), by extensionality.

For the right-to-left direction, suppose \( [x]_R = [y]_R \). Since \( R \) is reflexive, \( Ryy \), so \( y \in [y]_R \). Thus also \( y \in [x]_R \) by the assumption that \( [x]_R = [y]_R \). So \( Rxy \). \( \square \)

**Example rel.4.** A nice example of equivalence relations comes from modular arithmetic. For any \( a \), \( b \), and \( n \in \mathbb{N} \), say that \( a \equiv_n b \) iff dividing \( a \) by \( n \) gives the same remainder as dividing \( b \) by \( n \). (Somewhat more symbolically: \( a \equiv_n b \) iff, for some \( k \in \mathbb{Z} \), \( a - b = kn \).) Now, \( \equiv_n \) is an equivalence relation, for any \( n \).

And there are exactly \( n \) distinct equivalence classes generated by \( \equiv_n \); that is, \( \mathbb{N}/\equiv_n \) has \( n \) elements. These are: the set of numbers divisible by \( n \) without remainder, i.e., \( [0]_{\equiv_n} \); the set of numbers divisible by \( n \) with remainder 1, i.e., \( [1]_{\equiv_n} \); \( \ldots \); and the set of numbers divisible by \( n \) with remainder \( n-1 \), i.e., \( [n-1]_{\equiv_n} \).

**Problem rel.1.** Show that \( \equiv_n \) is an equivalence relation, for any \( n \in \mathbb{N} \), and that \( \mathbb{N}/\equiv_n \) has exactly \( n \) members.