

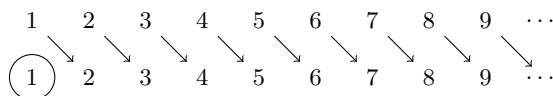
## infinite.1 Hilbert's Hotel

sfr:infinite:hilbert:  
sec

The set of the natural numbers is obviously infinite. So, if we do not want to *help ourselves* to the natural numbers, our first step must be characterize an infinite set in terms that do not require mentioning the natural numbers themselves. Here is a nice approach, presented by Hilbert in a lecture from 1924. He asks us to imagine

[...] a hotel with a finite number of rooms. All of these rooms should be occupied by exactly one guest. If the guests now swap their rooms somehow, [but] so that each room still contains no more than one person, then no rooms will become free, and the hotel-owner cannot in this way create a new place for a newly arriving guest [...]

Now we stipulate that the hotel shall have infinitely many numbered rooms 1, 2, 3, 4, 5, ..., each of which is occupied by exactly one guest. As soon as a new guest comes along, the owner only needs to move each of the old guests into the room associated with the number one higher, and room 1 will be free for the newly-arriving guest.



(published in [Hilbert 2013](#), 730; translation by Tim Button)

The crucial point is that Hilbert's Hotel has infinitely many rooms; and we can take his explanation to define what it means to say this. Indeed, this was Dedekind's approach (presented here, of course, with massive anachronism; Dedekind's definition is from [1888](#)):

sfr:infinite:hilbert:  
defn:DedekindInfinite

**Definition infinite.1.** A set  $A$  is *Dedekind infinite* iff there is an injection from  $A$  to a proper subset of  $A$ . That is, there is some  $o \in A$  and an injection  $f: A \rightarrow A$  such that  $o \notin \text{ran}(f)$ .

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## Bibliography

Dedekind, Richard. 1888. *Was sind und was sollen die Zahlen?* Braunschweig: Vieweg.

Hilbert, David. 2013. *David Hilbert's Lectures on the Foundations of Arithmetic and Logic 1917–1933*, eds. William Bragg Ewald and Wilfried Sieg. Heidelberg: Springer.