infinite.1 Dedekind Algebras and Arithmetical Induction

Crucially, now, a Dedekind algebra—indeed, any Dedekind algebra—will serve as a surrogate for the natural numbers. This is thanks to the following trivial consequence:

**Theorem infinite.1 (Arithmetical induction).** Let \( N, s, o \) comprise a Dedekind algebra. Then for any set \( X \):

\[
\text{if } o \in X \text{ and } (\forall n \in N \cap X)s(n) \in X, \text{ then } N \subseteq X.
\]

**Proof.** By the definition of a Dedekind algebra, \( N = \text{clo}_o(o) \). Now if both \( o \in X \) and \((\forall n \in N)(n \in X \rightarrow s(n) \in X)\), then \( N = \text{clo}_o(o) \subseteq X \).

Since induction is characteristic of the natural numbers, the point is this. Given any Dedekind infinite set, we can form a Dedekind algebra, and use that algebra as our surrogate for the natural numbers.

Admittedly, Theorem infinite.1 formulates induction in set-theoretic terms. But we can easily put the principle in terms which might be more familiar:

**Corollary infinite.2.** Let \( N, s, o \) comprise a Dedekind algebra. Then for any formula \( \varphi(x) \), which may have parameters:

\[
\text{if } \varphi(o) \text{ and } (\forall n \in N)(\varphi(n) \rightarrow \varphi(s(n))), \text{ then } (\forall n \in N)\varphi(n)
\]

**Proof.** Let \( X = \{n \in N : \varphi(n)\} \), and now use Theorem infinite.1

In this result, we spoke of a formula “having parameters”. What this means, roughly, is that for any objects \( c_1, \ldots, c_k \), we can work with \( \varphi(x, c_1, \ldots, c_k) \). More precisely, we can state the result without mentioning “parameters” as follows. For any formula \( \varphi(x, v_1, \ldots, v_k) \), whose free variables are all displayed, we have:

\[
\forall v_1 \ldots \forall v_k ((\varphi(o, v_1, \ldots, v_k) \land \\
(\forall x \in N)(\varphi(x, v_1, \ldots, v_k) \rightarrow \varphi(s(x), v_1, \ldots, v_k))) \rightarrow \\
(\forall x \in N)\varphi(x, v_1, \ldots, v_k))
\]

Evidently, speaking of “having parameters” can make things much easier to read. (In ??, we will use this device rather frequently.)

Returning to Dedekind algebras: given any Dedekind algebra, we can also define the usual arithmetical functions of addition, multiplication and exponentiation. This is non-trivial, however, and it involves the technique of recursive definition. That is a technique which we shall introduce and justify much later, and in a much more general context. (Enthusiasts might want to revisit this.
after ??, or perhaps read an alternative treatment, such as Potter 2004, pp. 95–8.) But, where $N, s, o$ comprise a Dedekind algebra, we will ultimately be able to stipulate the following:

$$
\begin{align*}
    a + o &= a & a \times o &= o & a^o &= s(a) \\
    a + s(b) &= s(a + b) & a \times s(b) &= (a \times b) + a & a^{s(b)} &= a^b \times a
\end{align*}
$$

and show that these behave as one would hope.

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Bibliography