

## infinite.1 Dedekind Algebras and Arithmetical Induction

sfr:infinite:induction:sec Crucially, now, a Dedekind algebra—indeed, *any* Dedekind algebra—will serve as a surrogate for the natural numbers. This is thanks to the following trivial consequence:

sfr:infinite:induction:thm:dedinfiniteinduction **Theorem infinite.1 (Arithmetical induction).** *Let  $N, s, o$  comprise a Dedekind algebra. Then for any set  $X$ :*

$$\text{if } o \in X \text{ and } (\forall n \in N \cap X)s(n) \in X, \text{ then } N \subseteq X.$$

*Proof.* By the definition of a Dedekind algebra,  $N = \text{clo}_s(o)$ . Now if both  $o \in X$  and  $(\forall n \in N)(n \in X \rightarrow s(n) \in X)$ , then  $N = \text{clo}_s(o) \subseteq X$ .  $\square$

Since induction is characteristic of the natural numbers, the point is this. Given any Dedekind infinite set, we can form a Dedekind algebra, and use that algebra as our surrogate for the natural numbers.

Admittedly, **Theorem infinite.1** formulates induction in *set-theoretic* terms. But we can easily put the principle in terms which might be more familiar:

sfr:infinite:induction:natinductionschem **Corollary infinite.2.** *Let  $N, s, o$  comprise a Dedekind algebra. Then for any formula  $\varphi(x)$ , which may have parameters:*

$$\text{if } \varphi(o) \text{ and } (\forall n \in N)(\varphi(n) \rightarrow \varphi(s(n))), \text{ then } (\forall n \in N)\varphi(n)$$

*Proof.* Let  $X = \{n \in N : \varphi(n)\}$ , and now use **Theorem infinite.1**  $\square$

In this result, we spoke of a formula “having parameters”. What this means, roughly, is that for any objects  $c_1, \dots, c_k$ , we can work with  $\varphi(x, c_1, \dots, c_k)$ . More precisely, we can state the result without mentioning “parameters” as follows. For any formula  $\varphi(x, v_1, \dots, v_k)$ , whose free variables are all displayed, we have:

$$\begin{aligned} \forall v_1 \dots \forall v_k ((\varphi(o, v_1, \dots, v_k) \wedge \\ (\forall x \in N)(\varphi(x, v_1, \dots, v_k) \rightarrow \varphi(s(x), v_1, \dots, v_k))) \rightarrow \\ (\forall x \in N)\varphi(x, v_1, \dots, v_k)) \end{aligned}$$

Evidently, speaking of “having parameters” can make things much easier to read. (In ??, we will use this device rather frequently.)

Returning to Dedekind algebras: given any Dedekind algebra, we can also define the usual arithmetical functions of addition, multiplication and exponentiation. This is non-trivial, however, and it involves the technique of *recursive definition*. That is a technique which we shall introduce and justify much later, and in a much more general context. (Enthusiasts might want to revisit this

after ??, or perhaps read an alternative treatment, such as [Potter 2004](#), pp. 95–8.) But, where  $N, s, o$  comprise a Dedekind algebra, we will ultimately be able to stipulate the following:

$$\begin{array}{lll} a + o = a & a \times o = o & a^o = s(o) \\ a + s(b) = s(a + b) & a \times s(b) = (a \times b) + a & a^{s(b)} = a^b \times a \end{array}$$

and show that these behave as one would hope.

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## Bibliography

Potter, Michael. 2004. *Set Theory and its Philosophy*. Oxford: Oxford University Press.