infinite.1 Dedekind Algebras and Arithmetical Induction

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Crucially, now, a Dedekind algebra—indeed, *any* Dedekind algebra—will serve as a surrogate for the natural numbers. This is thanks to the following trivial consequence:

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Theorem infinite.1 (Arithmetical induction). *Let N, s, o yield a Dedekind algebra. Then for any set X:*

$$\textit{if } o \in X \textit{ and } (\forall n \in N \cap X)s(n) \in X, \textit{ then } N \subseteq X.$$

Proof. By the definition of a Dedekind algebra, $N = \text{clo}_s(o)$. Now if both $o \in X$ and $(\forall n \in N)(n \in X \to s(n) \in X)$, then $N = \text{clo}_s(o) \subseteq X$. □

Since induction is characteristic of the natural numbers, the point is this. Given any Dedekind infinite set, we can form a Dedekind algebra, and use that algebra as our surrogate for the natural numbers.

Admittedly, Theorem infinite.1 formulates induction in *set-theoretic* terms. But we can easily put the principle in terms which might be more familiar:

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Corollary infinite.2. *Let N, s, o yield a Dedekind algebra. Then for any formula $\varphi(x)$, which may have parameters:*

$$\textit{if } \varphi(o) \textit{ and } (\forall n \in N)(\varphi(n) \to \varphi(s(n))), \textit{ then } (\forall n \in N)\varphi(n)$$

Proof. Let $X = \{n \in N : \varphi(n)\}$, and now use Theorem infinite.1 □

In this result, we spoke of a formula "having parameters". What this means, roughly, is that for any objects $c_1, \ldots, c_n$, we can work with $\varphi(x, c_1, \ldots, c_k)$. More precisely, we can state the result without mentioning "parameters" as follows. For any formula $\varphi(x, v_1, \ldots, v_k)$, whose free variables are all displayed, we have:

$$\begin{aligned}\forall v_1 \ldots \forall v_k ((\varphi(o, v_1, \ldots, v_k) \land{} \\ (\forall x \in N)(\varphi(x, v_1, \ldots, v_k) \to \varphi(s(x), v_1, \ldots, v_k))) \to{} \\ (\forall x \in N)\varphi(x, v_1, \ldots, v_k))\end{aligned}$$

Evidently, speaking of "having parameters" can make things much easier to read. (In **??**, we will use this device rather frequently.)

Returning to Dedekind algebras: given any Dedekind algebra, we can also define the usual arithmetical functions of addition, multiplication and exponentiation. This is non-trivial, however, and it involves the technique of *recursive definition*. That is a technique which we shall introduce and justify much later, and in a much more general context. (Enthusiasts might want to revisit this after **??**, or perhaps read an alternative treatment, such as Potter 2004, pp. 95–8.) But, where N, s, o yield a Dedekind algebra, we will ultimately be able to stipulate the following:

$$\begin{array}{ccc} m + o = m & m \times o = o & m^{o} = s(o) \\ m + s(n) = s(m + n) & m \times s(n) = (m \times n) + m & m^{s(n)} = m^{n} \times m \end{array}$$

and show that these behave as one would hope.

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Bibliography

Potter, Michael. 2004. *Set Theory and its Philosophy*. Oxford: Oxford University Press.