infinite.1 Dedekind Algebras

sfr:infinite:dedekind:sec

We not only want natural numbers to be infinite; we want them to have certain (algebraic) properties: they need to behave well under addition, multiplication, and so forth.

Dedekind's idea was to take the idea of the *successor function* as basic, and then characterise the numbers as those with the following properties:

1. There is a number, 0, which is not the successor of any number
   i.e., $0 \notin \operatorname{ran}(s)$
   i.e., $\forall x\ s(x) \neq 0$

2. Distinct numbers have distinct successors
   i.e., s is an injection
   i.e., $\forall x \forall y (s(x) = s(y) \rightarrow x = y)$

sfr:infinite:dedekind:repeatedapplication

3. Every number is obtained from 0 by repeated applications of the successor function.

The first two conditions are easy to deal with using first-order logic (see above). But we cannot deal with (3) just using first-order logic. Dedekind's breakthrough was to reformulate condition (3), set-theoretically, as follows:

3′. The natural numbers are the smallest set that is *closed under the successor function*: that is, if we apply s to any element of the set, we obtain another element of the set.

But we shall need to spell this out slowly.

sfr:infinite:dedekind:Closure

Definition infinite.1. For any function f, the set X is *f-closed* iff $(\forall x \in X) f(x) \in X$. Now define, for any o:

$$\operatorname{clo}_f(o) = \bigcap\{X : o \in X \text{ and } X \text{ is } f\text{-closed}\}$$

So $\operatorname{clo}_f(o)$ is the intersection of all the f-closed sets with o as an element. Intuitively, then, $\operatorname{clo}_f(o)$ is the *smallest* f-closed set with o as an element. This next result makes that intuitive thought precise;

sfr:infinite:dedekind:closureproperties

Lemma infinite.2. *For any function f and any $o \in A$:*

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1. *$o \in \operatorname{clo}_f(o)$; and*

sfr:infinite:dedekind:closureclosed

2. *$\operatorname{clo}_f(o)$ is f-closed; and*

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3. *if X is f-closed and $o \in X$, then $\operatorname{clo}_f(o) \subseteq X$*

Proof. Note that there is at least one f-closed set with o as an element, namely $\operatorname{ran}(f) \cup \{o\}$. So $\operatorname{clo}_f(o)$, the intersection of *all* such sets, exists. We must now check (1)–(3).

Concerning (1): $o \in \text{clo}_f(o)$ as it is an intersection of sets which all have o as an element.

Concerning (2): suppose $x \in \text{clo}_f(o)$. So if $o \in X$ and X is f-closed, then $x \in X$, and now $f(x) \in X$ as X is f-closed. So $f(x) \in \text{clo}_f(o)$.

Concerning (3): quite generally, if $X \in C$ then $\bigcap C \subseteq X$. □

Using this, we can say:

Definition infinite.3. A *Dedekind algebra* is a set A together with a function $f\colon A \to A$ and some $o \in A$ such that:

1. $o \notin \text{ran}(f)$ sfr:infinite:dedekind:ded:proper
2. f is an injection sfr:infinite:dedekind:ded:injection
3. $A = \text{clo}_f(o)$ sfr:infinite:dedekind:ded:closure

Since $A = \text{clo}_f(o)$, our earlier result tells us that A is the smallest f-closed set with o as an element. Clearly a Dedekind algebra is Dedekind infinite; just look at clauses (1) and (2) of the definition. But the more exciting fact is that any Dedekind infinite set can be turned into a Dedekind algebra.

Theorem infinite.4. *If there is a Dedekind infinite set, then there is a Dedekind algebra.* *sfr:infinite:dedekind:thm:DedekindInfiniteAlgebra*

Proof. Let D be Dedekind infinite. So there is an injection $g\colon D \to D$ and an element $o \in D \setminus \text{ran}(g)$. Now let $A = \text{clo}_g(o)$; by Lemma infinite.2, A exists and $o \in A$. Let $f = g{\restriction}_A$. We will show that A, f, o comprise a Dedekind algebra.

Concerning (1): $o \notin \text{ran}(g)$ and $\text{ran}(f) \subseteq \text{ran}(g)$ so $o \notin \text{ran}(f)$.

Concerning (2): g is an injection on D; so $f \subseteq g$ must be an injection.

Concerning (3): by Lemma infinite.2, A is g-closed; a fortiori, A is f-closed. So $\text{clo}_f(o) \subseteq A$ by Lemma infinite.2. Since also $\text{clo}_f(o)$ is f-closed and $f = g{\restriction}_A$, it follows that $\text{clo}_f(o)$ is g-closed. So $A \subseteq \text{clo}_f(o)$ by Lemma infinite.2. □

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Bibliography