Before we depart from naïve set theory, we have one last naïve (but sophisticated!) proof to consider. This is a proof of Schröder-Bernstein (??): if \( A \preceq B \) and \( B \preceq A \) then \( A \approx B \); i.e., given injections \( f : A \to B \) and \( g : B \to A \) there is a bijection \( h : A \to B \).

In this chapter, we followed Dedekind's notion of closures. In fact, Dedekind provided a lovely proof of using this notion, and we will present it here. The proof closely follows Potter (2004, pp. 157–8), if you want a slightly different but essentially similar treatment. A little googling will also convince you that this is a theorem—rather like the irrationality of \( \sqrt{2} \)—for which many interesting and different proofs exist.

Using similar notation as ??, let
\[
\text{Clo}_f(B) = \bigcap\{X : B \subseteq X \text{ and } X \text{ is } f\text{-closed}\}
\]
for each set \( B \) and function \( f \). Defined thus, \( \text{Clo}_f(B) \) is the smallest \( f\)-closed set containing \( B \), in that:

**Proposition infinite.1.** For any function \( f \), and any \( B \):

1. \( B \subseteq \text{Clo}_f(B) \); and
2. \( \text{Clo}_f(B) \) is \( f\)-closed; and
3. if \( X \) is \( f\)-closed and \( B \subseteq X \), then \( \text{Clo}_f(B) \subseteq X \).

**Proof.** Exactly as in ??.

We need one last fact to get to Bernstein:

**Proposition infinite.2.** If \( A \subseteq B \subseteq C \) and \( A \approx C \), then \( A \approx B \approx C \).

**Proof.** Given a bijection \( f : C \to A \), let \( F = \text{Clo}_f(C \setminus B) \) and define a function \( g \) with domain \( C \) as follows:
\[
g(x) = \begin{cases} 
  f(x) & \text{if } x \in F \\
  x & \text{otherwise}
\end{cases}
\]

We’ll show that \( g \) is a bijection from \( C \to B \), from which it will follow that \( g \circ f^{-1} : A \to B \) is a bijection, completing the proof.

First we claim that if \( x \in F \) but \( y \notin F \) then \( g(x) \neq g(y) \). For reductio suppose otherwise, so that \( y = g(y) = g(x) = f(x) \). Since \( x \in F \) and \( F \) is \( f\)-closed by Proposition infinite.1, we have \( y = f(x) \in F \), a contradiction.

Now suppose \( g(x) = g(y) \). So, by the above, \( x \in F \) iff \( y \in F \). If \( x, y \in F \), then \( f(x) = g(x) = g(y) = f(y) \) so that \( x = y \) since \( f \) is a bijection. If \( x, y \notin F \), then \( x = g(x) = g(y) = y \). So \( g \) is an injection.

It remains to show that \( \text{ran}(g) = B \). So fix \( x \in B \subseteq C \). If \( x \notin F \), then \( g(x) = x \). If \( x \in F \), then \( x = f(y) \) for some \( y \in F \), since \( x \in B \) and \( F \) is the smallest \( f\)-closed set extending \( C \setminus B \), so that \( g(y) = f(y) = x \).
Finally, here is the proof of the main result. Recall that given a function $h$ and set $D$, we define $h[D] = \{h(x) : x \in D\}$.

Proof of Schröder-Berstein. Let $f: A \to B$ and $g: B \to A$ be injections. Since $f[A] \subseteq B$ we have that $g[f[A]] \subseteq g[B] \subseteq A$. Also, $g \circ f: A \to g[f[A]]$ is an injection since both $g$ and $f$ are; and indeed $g \circ f$ is a bijection, just by the way we defined its codomain. So $A \approx g[f[A]]$, and hence by Proposition infinite.2 there is a bijection $h: A \to g[B]$. Moreover, $g^{-1}$ is a bijection $g[B] \to B$. So $g^{-1} \circ h: A \to B$ is a bijection. \qed

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Bibliography