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infinite.1 A Proof of Schröder-Bernstein

Before we depart from naïve set theory, we have one last naïve (but sophisticated!) proof to consider. This is a proof of Schröder-Bernstein (**??**): if $A \preceq B$ and $B \preceq A$ then $A \approx B$; i.e., given injections $f\colon A \to B$ and $g\colon B \to A$ there is a bijection $h\colon A \to B$.

In this chapter, we followed Dedekind's notion of *closures.* In fact, Dedekind provided a lovely proof of using this notion, and we will present it here. The proof closely follows Potter (2004, pp. 157–8), if you want a slightly different but essentially similar treatment. A little googling will also convince you that this is a theorem—rather like the irrationality of $\sqrt{2}$—for which *many* interesting and different proofs exist.

Using similar notation as **??**, let

$$\mathrm{Clo}_f(B) = \bigcap \{X : B \subseteq X \text{ and } X \text{ is } f\text{-closed}\}$$

for each set B and function f. Defined thus, $\mathrm{Clo}_f(B)$ is the smallest f-closed set containing B, in that:

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Proposition infinite.1. *For any function f, and any B:*

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1. *$B \subseteq \mathrm{Clo}_f(B)$; and*

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2. *$\mathrm{Clo}_f(B)$ is f-closed; and*

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3. *if X is f-closed and $B \subseteq X$, then $\mathrm{Clo}_f(B) \subseteq X$.*

Proof. Exactly as in **??**. □

We need one last fact to get to Bernstein:

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Proposition infinite.2. *If $A \subseteq B \subseteq C$ and $A \approx C$, then $A \approx B \approx C$.*

Proof. Given a bijection $f\colon C \to A$, let $F = \mathrm{Clo}_f(C \setminus B)$ and define a function g with domain C as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \in F \\ x & \text{otherwise} \end{cases}$$

We'll show that g is a bijection from $C \to B$, from which it will follow that $g \circ f^{-1}\colon A \to B$ is a bijection, completing the proof.

First we claim that if $x \in F$ but $y \notin F$ then $g(x) \neq g(y)$. For reductio suppose otherwise, so that $y = g(y) = g(x) = f(x)$. Since $x \in F$ and F is f-closed by Proposition infinite.1, we have $y = f(x) \in F$, a contradiction.

Now suppose $g(x) = g(y)$. So, by the above, $x \in F$ iff $y \in F$. If $x, y \in F$, then $f(x) = g(x) = g(y) = f(y)$ so that $x = y$ since f is a bijection. If $x, y \notin F$, then $x = g(x) = g(y) = y$. So g is an injection.

It remains to show that $\mathrm{ran}(g) = B$. So fix $x \in B \subseteq C$. If $x \notin F$, then $g(x) = x$. If $x \in F$, then $x = f(y)$ for some $y \in F$, since $x \in B$ and F is the *smallest* f-closed set extending $C \setminus B$, so that $g(y) = f(y) = x$. □

Finally, here is the proof of the main result. Recall that given a function h and set D, we define $h[D] = \{h(x) : x \in D\}$.

Proof of Schröder-Berstein.. Let $f\colon A \to B$ and $g\colon B \to A$ be injections. Since $f[A] \subseteq B$ we have that $g[f[A]] \subseteq g[B] \subseteq A$. Also, $g \circ f\colon A \to g[f[A]]$ is an injection since both g and f are; and indeed $g \circ f$ is a bijection, just by the way we defined its codomain. So $A \approx g[f[A]]$, and hence by Proposition infinite.2 there is a bijection $h\colon A \to g[B]$. Moreover, g^{-1} is a bijection $g[B] \to B$. So $g^{-1} \circ h\colon A \to B$ is a bijection. □

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Bibliography

Potter, Michael. 2004. *Set Theory and its Philosophy*. Oxford: Oxford University Press.