fun.1 Partial Functions

sfr:fun:par:sec It is sometimes useful to relax the definition of function so that it is not required that the output of the function is defined for all possible inputs. Such mappings are called *partial functions*. explanation

Definition fun.1. A *partial function* $f \colon A \nrightarrow B$ is a mapping which assigns to every element of A at most one element of B. If f assigns an element of B to $x \in A$, we say $f(x)$ is *defined*, and otherwise *undefined*. If $f(x)$ is defined, we write $f(x) \downarrow$, otherwise $f(x) \uparrow$. The *domain* of a partial function f is the subset of A where it is defined, i.e., $\operatorname{dom}(f) = \{x \in A : f(x) \downarrow\}$.

Example fun.2. Every function $f \colon A \to B$ is also a partial function. Partial functions that are defined everywhere on A—i.e., what we so far have simply called a function—are also called *total* functions.

Example fun.3. The partial function $f \colon \mathbb{R} \nrightarrow \mathbb{R}$ given by $f(x) = 1/x$ is undefined for $x = 0$, and defined everywhere else.

Problem fun.1. Given $f \colon A \nrightarrow B$, define the partial function $g \colon B \nrightarrow A$ by: for any $y \in B$, if there is a unique $x \in A$ such that $f(x) = y$, then $g(y) = x$; otherwise $g(y) \uparrow$. Show that if f is injective, then $g(f(x)) = x$ for all $x \in \operatorname{dom}(f)$, and $f(g(y)) = y$ for all $y \in \operatorname{ran}(f)$.

Definition fun.4 (Graph of a partial function). Let $f \colon A \nrightarrow B$ be a partial function. The *graph* of f is the relation $R_f \subseteq A \times B$ defined by

$$R_f = \{\langle x, y \rangle : f(x) = y\}.$$

Proposition fun.5. *Suppose $R \subseteq A \times B$ has the property that whenever Rxy and Rxy' then $y = y'$. Then R is the graph of the partial function $f \colon X \nrightarrow Y$ defined by: if there is a y such that Rxy, then $f(x) = y$, otherwise $f(x) \uparrow$. If R is also* serial*, i.e., for each $x \in X$ there is a $y \in Y$ such that Rxy, then f is total.*

Proof. Suppose there is a y such that Rxy. If there were another $y' \neq y$ such that Rxy', the condition on R would be violated. Hence, if there is a y such that Rxy, that y is unique, and so f is well-defined. Obviously, $R_f = R$ and f is total if R is serial. □

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Bibliography