fun.1 Inverses of Functions

One obvious question about functions is whether a given mapping can be “reversed.” For instance, the successor function \( f(x) = x + 1 \) can be reversed in the sense that the function \( g(y) = y - 1 \) “undoes” what \( f \) does. But we must be careful: While the definition of \( g \) defines a function \( \mathbb{Z} \to \mathbb{Z} \), it does not define a function \( \mathbb{N} \to \mathbb{N} \) (\( g(0) \notin \mathbb{N} \)). So even in simple cases, it is not quite obvious if functions can be reversed, and that it may depend on the domain and codomain. Let’s give a precise definition.

**Definition fun.1.** A function \( g: Y \to X \) is an inverse of a function \( f: X \to Y \) if \( f(g(y)) = y \) and \( g(f(x)) = x \) for all \( x \in X \) and \( y \in Y \).

When do functions have inverses? A good candidate for an inverse of \( f: X \to Y \) is \( g: Y \to X \) “defined by”

\[
g(y) = \text{“the” } x \text{ such that } f(x) = y.
\]

The scare quotes around “defined by” suggest that this is not a definition. At least, it is not in general. For in order for this definition to specify a function, there has to be one and only one \( x \) such that \( f(x) = y \)—the output of \( g \) has to be uniquely specified. Moreover, it has to be specified for every \( y \in Y \). If there are \( x_1 \) and \( x_2 \in X \) with \( x_1 \neq x_2 \) but \( f(x_1) = f(x_2) \), then \( g(y) \) would not be uniquely specified for \( y = f(x_1) = f(x_2) \). And if there is no \( x \) at all such that \( f(x) = y \), then \( g(y) \) is not specified at all. In other words, for \( g \) to be defined, \( f \) has to be injective and surjective.

**Proposition fun.2.** If \( f: X \to Y \) is bijective, \( f \) has a unique inverse \( f^{-1}: Y \to X \).

**Proof.** Exercise. \( \square \)

**Problem fun.1.** Show that if \( f \) is bijective, an inverse \( g \) of \( f \) exists, i.e., define such a \( g \), show that it is a function, and show that it is an inverse of \( f \), i.e., \( f(g(y)) = y \) and \( g(f(x)) = x \) for all \( x \in X \) and \( y \in Y \).

**Problem fun.2.** Show that if \( f: X \to Y \) has an inverse \( g \), then \( f \) is bijective.

**Problem fun.3.** Show that if \( g: Y \to X \) and \( g': Y \to X \) are inverses of \( f: X \to Y \), then \( g = g' \), i.e., for all \( y \in Y \), \( g(y) = g'(y) \).

Photo Credits

Bibliography