fun.1 Inverses of Functions

sfr:fun:inv:sec

We think of functions as maps. An obvious question to ask about functions, then, is whether the mapping can be "reversed." For instance, the successor function $f(x) = x+1$ can be reversed, in the sense that the function $g(y) = y-1$ "undoes" what f does. explanation

But we must be careful. Although the definition of g defines a function $\mathbb{Z} \to \mathbb{Z}$, it does not define a *function* $\mathbb{N} \to \mathbb{N}$, since $g(0) \notin \mathbb{N}$. So even in simple cases, it is not quite obvious whether a function can be reversed; it may depend on the domain and codomain.

This is made more precise by the notion of an inverse of a function.

Definition fun.1. A function $g\colon B \to A$ is an *inverse* of a function $f\colon A \to B$ if $f(g(y)) = y$ and $g(f(x)) = x$ for all $x \in A$ and $y \in B$.

If f has an inverse g, we often write f^{-1} instead of g.

Now we will determine when functions have inverses. A good candidate for an inverse of $f\colon A \to B$ is $g\colon B \to A$ "defined by" explanation

$$g(y) = \text{"the" } x \text{ such that } f(x) = y.$$

But the scare quotes around "defined by" (and "the") suggest that this is not a definition. At least, it will not always work, with complete generality. For, in order for this definition to specify a function, there has to be one and only one x such that $f(x) = y$—the output of g has to be uniquely specified. Moreover, it has to be specified for every $y \in B$. If there are x_1 and $x_2 \in A$ with $x_1 \neq x_2$ but $f(x_1) = f(x_2)$, then $g(y)$ would not be uniquely specified for $y = f(x_1) = f(x_2)$. And if there is no x at all such that $f(x) = y$, then $g(y)$ is not specified at all. In other words, for g to be defined, f must be both injective and surjective.

Let's go slowly. We'll divide the question into two: Given a function $f\colon A \to B$, when is there a function $g\colon B \to A$ so that $g(f(x)) = x$? Such a g "undoes" what f does, and is called a *left inverse* of f. Secondly, when is there a function $h\colon B \to A$ so that $f(h(y)) = y$? Such an h is called a *right inverse* of f—f "undoes" what h does.

Proposition fun.2. *If $f\colon A \to B$ is injective, then there is a* left inverse *$g\colon B \to A$ of f so that $g(f(x)) = x$ for all $x \in A$.*

Proof. Suppose that $f\colon A \to B$ is injective. Consider a $y \in B$. If $y \in \operatorname{ran}(f)$, there is an $x \in A$ so that $f(x) = y$. Because f is injective, there is only one such $x \in A$. Then we can define: $g(y) = x$, i.e., $g(y)$ is "the" $x \in A$ such that $f(x) = y$. If $y \notin \operatorname{ran}(f)$, we can map it to any $a \in A$. So, we can pick an $a \in A$ and define $g\colon B \to A$ by:

$$g(y) = \begin{cases} x & \text{if } f(x) = y \\ a & \text{if } y \notin \operatorname{ran}(f). \end{cases}$$

It is defined for all $y \in B$, since for each such $y \in \operatorname{ran}(f)$ there is exactly one $x \in A$ such that $f(x) = y$. By definition, if $y = f(x)$, then $g(y) = x$, i.e., $g(f(x)) = x$. □

Problem fun.1. Show that if $f\colon A \to B$ has a left inverse g, then f is injective.

Proposition fun.3. *If $f\colon A \to B$ is surjective, then there is a* right inverse *$h\colon B \to A$ of f so that $f(h(y)) = y$ for all $y \in B$.*

Proof. Suppose that $f\colon A \to B$ is surjective. Consider a $y \in B$. Since f is surjective, there is an $x_y \in A$ with $f(x_y) = y$. Then we can define: $h(y) = x_y$, i.e., for each $y \in B$ we choose some $x \in A$ so that $f(x) = y$; since f is surjective there is always at least one to choose from.[1] By definition, if $x = h(y)$, then $f(x) = y$, i.e., for any $y \in B$, $f(h(y)) = y$. □

Problem fun.2. Show that if $f\colon A \to B$ has a right inverse h, then f is surjective.

explanation

By combining the ideas in the previous proof, we now get that every bijection has an inverse, i.e., there is a single function which is both a left and right inverse of f.

Proposition fun.4. *If $f\colon A \to B$ is bijective, there is a function $f^{-1}\colon B \to A$ so that for all $x \in A$, $f^{-1}(f(x)) = x$ and for all $y \in B$, $f(f^{-1}(y)) = y$.* *sfr:fun:inv:prop:bijection-inverse*

Proof. Exercise. □

Problem fun.3. Prove Proposition fun.4. You have to define f^{-1}, show that it is a function, and show that it is an inverse of f, i.e., $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$ for all $x \in A$ and $y \in B$.

explanation

There is a slightly more general way to extract inverses. We saw in **??** that every function f induces a surjection $f'\colon A \to \operatorname{ran}(f)$ by letting $f'(x) = f(x)$ for all $x \in A$. Clearly, if f is injective, then f' is bijective, so that it has a unique inverse by Proposition fun.4. By a very minor abuse of notation, we sometimes call the inverse of f' simply "the inverse of f."

Proposition fun.5. *Show that if $f\colon A \to B$ has a left inverse g and a right inverse h, then $h = g$.* *sfr:fun:inv:prop:left-right*

Proof. Exercise. □

[1] Since f is surjective, for every $y \in B$ the set $\{x : f(x) = y\}$ is nonempty. Our definition of h requires that we choose a single x from each of these sets. That this is always possible is actually not obvious—the possibility of making these choices is simply assumed as an axiom. In other words, this proposition assumes the so-called Axiom of Choice, an issue we will gloss over. However, in many specific cases, e.g., when $A = \mathbb{N}$ or is finite, or when f is bijective, the Axiom of Choice is not required. (In the particular case when f is bijective, for each $y \in B$ the set $\{x : f(x) = y\}$ has exactly one element, so that there is no choice to make.)

Problem fun.4. Prove Proposition fun.5.

sfr:fun:inv: prop:inverse-unique **Proposition fun.6.** *Every function f has at most one inverse.*

Proof. Suppose g and h are both inverses of f. Then in particular g is a left inverse of f and h is a right inverse. By Proposition fun.5, $g = h$. $\square$

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Bibliography