fun.1 Inverses of Functions

We think of functions as maps. An obvious question to ask about functions, then, is whether the mapping can be “reversed.” For instance, the successor function $f(x) = x + 1$ can be reversed, in the sense that the function $g(y) = y - 1$ “undoes” what $f$ does.

But we must be careful. Although the definition of $g$ defines a function $\mathbb{Z} \to \mathbb{Z}$, it does not define a function $\mathbb{N} \to \mathbb{N}$, since $g(0) \notin \mathbb{N}$. So even in simple cases, it is not quite obvious whether a function can be reversed; it may depend on the domain and codomain.

This is made more precise by the notion of an inverse of a function.

**Definition fun.1.** A function $g: B \to A$ is an inverse of a function $f: A \to B$ if $f(g(y)) = y$ and $g(f(x)) = x$ for all $x \in A$ and $y \in B$.

If $f$ has an inverse $g$, we often write $f^{-1}$ instead of $g$.

Now we will determine when functions have inverses. A good candidate for an inverse of $f: A \to B$ is $g: B \to A$ “defined by” $g(y) = “the” \ x \ such \ that \ f(x) = y$.

But the scare quotes around “defined by” (and “the”) suggest that this is not a definition. At least, it will not always work, with complete generality. For, in order for this definition to specify a function, there has to be one and only one $x$ such that $f(x) = y$—the output of $g$ has to be uniquely specified. Moreover, it has to be specified for every $y \in B$. If there are $x_1$ and $x_2 \in A$ with $x_1 \neq x_2$ but $f(x_1) = f(x_2)$, then $g(y)$ would not be uniquely specified for $y = f(x_1) = f(x_2)$. And if there is no $x$ at all such that $f(x) = y$, then $g(y)$ is not specified at all. In other words, for $g$ to be defined, $f$ must be both injective and surjective.

**Proposition fun.2.** Every bijection has a unique inverse.

*Proof. Exercise.*

**Problem fun.1.** Prove Proposition fun.2. That is, show that if $f: A \to B$ is bijective, an inverse $g$ of $f$ exists. You have to define such a $g$, show that it is a function, and show that it is an inverse of $f$, i.e., $f(g(y)) = y$ and $g(f(x)) = x$ for all $x \in A$ and $y \in B$.

However, there is a slightly more general way to extract inverses. We saw in ?? that every function $f$ induces a surjection $f': A \to \text{ran}(f)$ by letting $f'(x) = f(x)$ for all $x \in A$. Clearly, if $f$ is an injection, then $f'$ is a bijection, so that it has a unique inverse by Proposition fun.2. By a very minor abuse of notation, we sometimes call the inverse of $f'$ simply “the inverse of $f$.”

**Problem fun.2.** Show that if $f: A \to B$ has an inverse $g$, then $f$ is bijective.

**Proposition fun.3.** Every function $f$ has at most one inverse.
Proof. Exercise. \qed

Problem fun.3. Prove Proposition fun.3. That is, show that if \(g: B \to A\) and \(g': B \to A\) are inverses of \(f: A \to B\), then \(g = g'\), i.e., for all \(y \in B\), \(g(y) = g'(y)\).

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Bibliography