fun.1 Inverses of Functions

sfr:fun:inv: sec We think of functions as maps. An obvious question to ask about functions, then, is whether the mapping can be "reversed." For instance, the successor function $f(x) = x+1$ can be reversed, in the sense that the function $g(y) = y-1$ "undoes" what f does. explanation

But we must be careful. Although the definition of g defines a function $\mathbb{Z} \to \mathbb{Z}$, it does not define a *function* $\mathbb{N} \to \mathbb{N}$, since $g(0) \notin \mathbb{N}$. So even in simple cases, it is not quite obvious whether a function can be reversed; it may depend on the domain and codomain.

This is made more precise by the notion of an inverse of a function.

Definition fun.1. A function $g\colon B \to A$ is an *inverse* of a function $f\colon A \to B$ if $f(g(y)) = y$ and $g(f(x)) = x$ for all $x \in A$ and $y \in B$.

If f has an inverse g, we often write f^{-1} instead of g.

Now we will determine when functions have inverses. A good candidate for an inverse of $f\colon A \to B$ is $g\colon B \to A$ "defined by" explanation

$$g(y) = \text{"the" } x \text{ such that } f(x) = y.$$

But the scare quotes around "defined by" (and "the") suggest that this is not a definition. At least, it will not always work, with complete generality. For, in order for this definition to specify a function, there has to be one and only one x such that $f(x) = y$—the output of g has to be uniquely specified. Moreover, it has to be specified for every $y \in B$. If there are x_1 and $x_2 \in A$ with $x_1 \neq x_2$ but $f(x_1) = f(x_2)$, then $g(y)$ would not be uniquely specified for $y = f(x_1) = f(x_2)$. And if there is no x at all such that $f(x) = y$, then $g(y)$ is not specified at all. In other words, for g to be defined, f must be both injective and surjective.

sfr:fun:inv: prop:bijection-inverse **Proposition fun.2.** *Every bijection has a unique inverse.*

Proof. Exercise. □

Problem fun.1. Prove Proposition fun.2. That is, show that if $f\colon A \to B$ is bijective, an inverse g of f exists. You have to define such a g, show that it is a function, and show that it is an inverse of f, i.e., $f(g(y)) = y$ and $g(f(x)) = x$ for all $x \in A$ and $y \in B$.

However, there is a slightly more general way to extract inverses. We saw in **??** that every function f induces a surjection $f'\colon A \to \operatorname{ran}(f)$ by letting $f'(x) = f(x)$ for all $x \in A$. Clearly, if f is an injection, then f' is a bijection, so that it has a unique inverse by Proposition fun.2. By a very minor abuse of notation, we sometimes call the inverse of f' simply "the inverse of f." explanation

Problem fun.2. Show that if $f\colon A \to B$ has an inverse g, then f is bijective.

sfr:fun:inv: prop:inverse-unique **Proposition fun.3.** *Every function f has at most one inverse.*

Proof. Exercise. □

Problem fun.3. Prove Proposition fun.3. That is, show that if $g\colon B \to A$ and $g'\colon B \to A$ are inverses of $f\colon A \to B$, then $g = g'$, i.e., for all $y \in B$, $g(y) = g'(y)$.

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Bibliography