# Chapter udf

# Functions

## fun.1 Basics

sfr:fun:bas:sec A *function* is a map which sends each element of a given set to a specific element in some (other) given set. For instance, the operation of adding 1 defines a function: each number $n$ is mapped to a unique number $n+1$. explanation

More generally, functions may take pairs, triples, etc., as inputs and returns some kind of output. Many functions are familiar to us from basic arithmetic. For instance, addition and multiplication are functions. They take in two numbers and return a third.

In this mathematical, abstract sense, a function is a *black box*: what matters is only what output is paired with what input, not the method for calculating the output.

**Definition fun.1 (Function).** A *function* $f\colon A \to B$ is a mapping of each element of $A$ to an element of $B$.

We call $A$ the *domain* of $f$ and $B$ the *codomain* of $f$. The elements of $A$ are called inputs or *arguments* of $f$, and the element of $B$ that is paired with an argument $x$ by $f$ is called the *value of $f$* for argument $x$, written $f(x)$.

The *range* $\operatorname{ran}(f)$ of $f$ is the subset of the codomain consisting of the values of $f$ for some argument; $\operatorname{ran}(f) = \{f(x) : x \in A\}$.

The diagram in Figure fun.1 may help to think about functions. The ellipse on the left represents the function's *domain*; the ellipse on the right represents the function's *codomain*; and an arrow points from an *argument* in the domain to the corresponding *value* in the codomain.

**Example fun.2.** Multiplication takes pairs of natural numbers as inputs and maps them to natural numbers as outputs, so goes from $\mathbb{N} \times \mathbb{N}$ (the domain) to $\mathbb{N}$ (the codomain). As it turns out, the range is also $\mathbb{N}$, since every $n \in \mathbb{N}$ is $n \times 1$.

**Example fun.3.** Multiplication is a function because it pairs each input—each pair of natural numbers—with a single output: $\times\colon \mathbb{N}^2 \to \mathbb{N}$. By contrast,

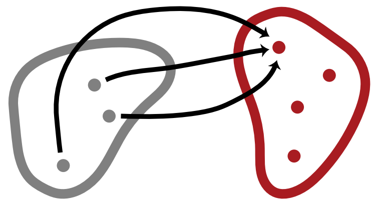

Figure fun.1: A function is a mapping of each element of one set to an element of another. An arrow points from an argument in the domain to the corresponding value in the codomain.

sfr:fun:bas:
fig:function

the square root operation applied to the domain $\mathbb{N}$ is not functional, since each positive integer $n$ has two square roots: $\sqrt{n}$ and $-\sqrt{n}$. We can make it functional by only returning the positive square root: $\sqrt{\phantom{x}} \colon \mathbb{N} \to \mathbb{R}$.

**Example fun.4.** The relation that pairs each student in a class with their final grade is a function—no student can get two different final grades in the same class. The relation that pairs each student in a class with their parents is not a function: students can have zero, or two, or more parents.

explanation

We can define functions by specifying in some precise way what the value of the function is for every possible argment. Different ways of doing this are by giving a formula, describing a method for computing the value, or listing the values for each argument. However functions are defined, we must make sure that for each argment we specify one, and only one, value.

**Example fun.5.** Let $f \colon \mathbb{N} \to \mathbb{N}$ be defined such that $f(x) = x + 1$. This is a definition that specifies $f$ as a function which takes in natural numbers and outputs natural numbers. It tells us that, given a natural number $x$, $f$ will output its successor $x + 1$. In this case, the codomain $\mathbb{N}$ is not the range of $f$, since the natural number 0 is not the successor of any natural number. The range of $f$ is the set of all positive integers, $\mathbb{Z}^+$.

**Example fun.6.** Let $g \colon \mathbb{N} \to \mathbb{N}$ be defined such that $g(x) = x + 2 - 1$. This tells us that $g$ is a function which takes in natural numbers and outputs natural numbers. Given a natural number $n$, $g$ will output the predecessor of the successor of the successor of $x$, i.e., $x + 1$.

sfr:fun:bas:
examplefunext

explanation

We just considered two functions, $f$ and $g$, with different *definitions*. However, these are the *same function*. After all, for any natural number $n$, we have that $f(n) = n + 1 = n + 2 - 1 = g(n)$. Otherwise put: our definitions for $f$ and $g$ specify the same mapping by means of different equations. Implicitly, then, we are relying upon a principle of extensionality for functions,

$$\text{if } \forall x\, f(x) = g(x), \text{ then } f = g$$

provided that $f$ and $g$ share the same domain and codomain.

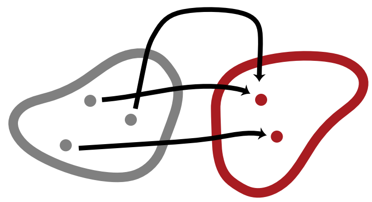

Figure fun.2: A surjective function has every element of the codomain as a value.

sfr:fun:kin:fig:surjective

**Example fun.7.** We can also define functions by cases. For instance, we could define $h\colon \mathbb{N} \to \mathbb{N}$ by

$$h(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{x+1}{2} & \text{if } x \text{ is odd.} \end{cases}$$

Since every natural number is either even or odd, the output of this function will always be a natural number. Just remember that if you define a function by cases, every possible input must fall into exactly one case. In some cases, this will require a proof that the cases are exhaustive and exclusive.

## fun.2 Kinds of Functions

sfr:fun:kin:sec

It will be useful to introduce a kind of taxonomy for some of the kinds of functions which we encounter most frequently. explanation

To start, we might want to consider functions which have the property that every member of the codomain is a value of the function. Such functions are called surjective, and can be pictured as in Figure fun.2.

**Definition fun.8 (Surjective function).** A function $f\colon A \to B$ is *surjective* iff $B$ is also the range of $f$, i.e., for every $y \in B$ there is at least one $x \in A$ such that $f(x) = y$, or in symbols:

$$(\forall y \in B)(\exists x \in A) f(x) = y.$$

We call such a function a surjection from $A$ to $B$.

If you want to show that $f$ is a surjection, then you need to show that every object in $f$'s codomain is the value of $f(x)$ for some input $x$. explanation

Note that any function *induces* a surjection. After all, given a function $f\colon A \to B$, let $f'\colon A \to \operatorname{ran}(f)$ be defined by $f'(x) = f(x)$. Since $\operatorname{ran}(f)$ is *defined* as $\{f(x) \in B : x \in A\}$, this function $f'$ is guaranteed to be a surjection

Now, any function maps each possible input to a unique output. But there are also functions which never map different inputs to the same outputs. Such functions are called injective, and can be pictured as in Figure fun.3. explanation

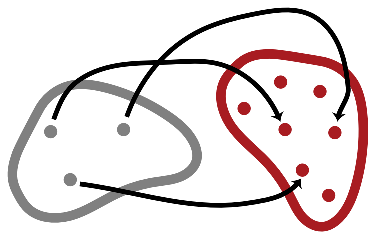

Figure fun.3: An injective function never maps two different arguments to the same value.

sfr:fun:kin:
fig:injective

**Definition fun.9 (Injective function).** A function $f\colon A \to B$ is *injective* iff for each $y \in B$ there is at most one $x \in A$ such that $f(x) = y$. We call such a function an injection from $A$ to $B$.

explanation

If you want to show that $f$ is an injection, you need to show that for any elements $x$ and $y$ of $f$'s domain, if $f(x) = f(y)$, then $x = y$.

**Example fun.10.** The constant function $f\colon \mathbb{N} \to \mathbb{N}$ given by $f(x) = 1$ is neither injective, nor surjective.

The identity function $f\colon \mathbb{N} \to \mathbb{N}$ given by $f(x) = x$ is both injective and surjective.

The successor function $f\colon \mathbb{N} \to \mathbb{N}$ given by $f(x) = x + 1$ is injective but not surjective.

The function $f\colon \mathbb{N} \to \mathbb{N}$ defined by:

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{x+1}{2} & \text{if } x \text{ is odd.} \end{cases}$$

is surjective, but not injective.

explanation

Often enough, we want to consider functions which are both injective and surjective. We call such functions bijective. They look like the function pictured in Figure fun.4. Bijections are also sometimes called *one-to-one correspondences*, since they uniquely pair elements of the codomain with elements of the domain.

**Definition fun.11 (Bijection).** A function $f\colon A \to B$ is *bijective* iff it is both surjective and injective. We call such a function a bijection from $A$ to $B$ (or between $A$ and $B$).

## fun.3 Functions as Relations

explanation

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sec

A function which maps elements of $A$ to elements of $B$ obviously defines a relation between $A$ and $B$, namely the relation which holds between $x$ and $y$ iff $f(x) = y$. In fact, we might even—if we are interested in reducing the building

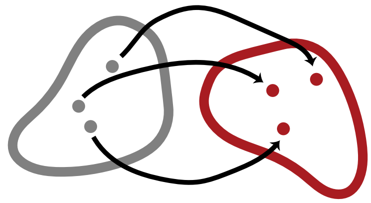

Figure fun.4: A bijective function uniquely pairs the elements of the codomain with those of the domain.

sfr:fun:kin:
fig:bijective

blocks of mathematics for instance—*identify* the function $f$ with this relation, i.e., with a set of pairs. This then raises the question: which relations define functions in this way?

**Definition fun.12 (Graph of a function).** Let $f\colon A \to B$ be a function. The *graph* of $f$ is the relation $R_f \subseteq A \times B$ defined by

$$R_f = \{\langle x, y\rangle : f(x) = y\}.$$

The graph of a function is uniquely determined, by extensionality. Moreover, extensionality (on sets) will immediate vindicate the implicit principle of extensionality for functions, whereby if $f$ and $g$ share a domain and codomain then they are identical if they agree on all values. explanation

Similarly, if a relation is "functional", then it is the graph of a function.

*sfr:fun:rel:*
*prop:graph-function*

**Proposition fun.13.** *Let $R \subseteq A \times B$ be such that:*

1. *If $Rxy$ and $Rxz$ then $y = z$; and*
2. *for every $x \in A$ there is some $y \in B$ such that $\langle x, y\rangle \in R$.*

*Then $R$ is the graph of the function $f\colon A \to B$ defined by $f(x) = y$ iff $Rxy$.*

*Proof.* Suppose there is a $y$ such that $Rxy$. If there were another $z \neq y$ such that $Rxz$, the condition on $R$ would be violated. Hence, if there is a $y$ such that $Rxy$, this $y$ is unique, and so $f$ is well-defined. Obviously, $R_f = R$. □

Every function $f\colon A \to B$ has a graph, i.e., a relation on $A \times B$ defined by $f(x) = y$. On the other hand, every relation $R \subseteq A \times B$ with the properties given in Proposition fun.13 is the graph of a function $f\colon A \to B$. Because of this close connection between functions and their graphs, we can think of a function simply as its graph. In other words, functions can be identified with certain relations, i.e., with certain sets of tuples. We can now consider performing similar operations on functions as we performed on relations (see **??**). In particular: explanation

**Definition fun.14.** Let $f\colon A \to B$ be a function with $C \subseteq A$. sfr:fun:rel: defn:funimage

The *restriction* of $f$ to $C$ is the function $f{\restriction_C}\colon C \to B$ defined by $(f{\restriction_C})(x) = f(x)$ for all $x \in C$. In other words, $f{\restriction_C} = \{\langle x, y\rangle \in R_f : x \in C\}$.

The *application* of $f$ to $C$ is $f[C] = \{f(x) : x \in C\}$. We also call this the *image* of $C$ under $f$.

explanation It follows from these definition that $\operatorname{ran}(f) = f[\operatorname{dom}(f)]$, for any function $f$.

## fun.4 Inverses of Functions

explanation We think of functions as maps. An obvious question to ask about functions, then, is whether the mapping can be "reversed." For instance, the successor function $f(x) = x+1$ can be reversed, in the sense that the function $g(y) = y-1$ "undoes" what $f$ does. sfr:fun:inv: sec

But we must be careful. Although the definition of $g$ defines a function $\mathbb{Z} \to \mathbb{Z}$, it does not define a *function* $\mathbb{N} \to \mathbb{N}$, since $g(0) \notin \mathbb{N}$. So even in simple cases, it is not quite obvious whether a function can be reversed; it may depend on the domain and codomain.

This is made more precise by the notion of an inverse of a function.

**Definition fun.15.** A function $g\colon B \to A$ is an *inverse* of a function $f\colon A \to B$ if $f(g(y)) = y$ and $g(f(x)) = x$ for all $x \in A$ and $y \in B$.

If $f$ has an inverse $g$, we often write $f^{-1}$ instead of $g$.

explanation Now we will determine when functions have inverses. A good candidate for an inverse of $f\colon A \to B$ is $g\colon B \to A$ "defined by"

$$g(y) = \text{“the” } x \text{ such that } f(x) = y.$$

But the scare quotes around "defined by" (and "the") suggest that this is not a definition. At least, it will not always work, with complete generality. For, in order for this definition to specify a function, there has to be one and only one $x$ such that $f(x) = y$—the output of $g$ has to be uniquely specified. Moreover, it has to be specified for every $y \in B$. If there are $x_1$ and $x_2 \in A$ with $x_1 \neq x_2$ but $f(x_1) = f(x_2)$, then $g(y)$ would not be uniquely specified for $y = f(x_1) = f(x_2)$. And if there is no $x$ at all such that $f(x) = y$, then $g(y)$ is not specified at all. In other words, for $g$ to be defined, $f$ must be both injective and surjective.

**Proposition fun.16.** *Every bijection has a unique inverse.* *sfr:fun:inv: prop:bijection-inverse*

*Proof.* Exercise. □

**Problem fun.1.** Prove Proposition fun.16. That is, show that if $f\colon A \to B$ is bijective, an inverse $g$ of $f$ exists. You have to define such a $g$, show that it is a function, and show that it is an inverse of $f$, i.e., $f(g(y)) = y$ and $g(f(x)) = x$ for all $x \in A$ and $y \in B$.

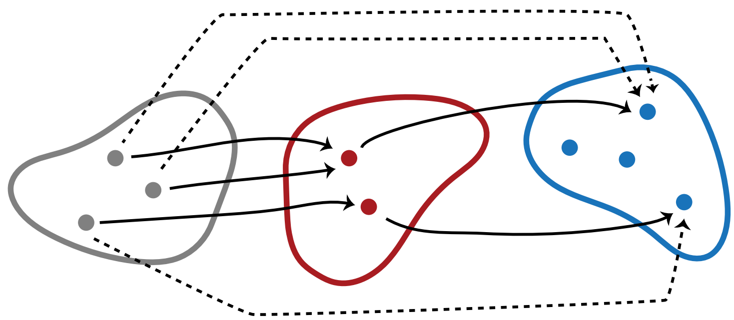

Figure fun.5: The composition $g \circ f$ of two functions $f$ and $g$.

sfr:fun:cmp:
fig:composition

However, there is a slightly more general way to extract inverses. We saw in section fun.2 that every function $f$ induces a surjection $f' : A \to \text{ran}(f)$ by letting $f'(x) = f(x)$ for all $x \in A$. Clearly, if $f$ is an injection, then $f'$ is a bijection, so that it has a unique inverse by Proposition fun.16. By a very minor abuse of notation, we sometimes call the inverse of $f'$ simply "the inverse of $f$." explanation

**Problem fun.2.** Show that if $f : A \to B$ has an inverse $g$, then $f$ is bijective.

*sfr:fun:inv: prop:inverse-unique*

**Proposition fun.17.** *Every function $f$ has at most one inverse.*

*Proof.* Exercise. □

**Problem fun.3.** Prove Proposition fun.17. That is, show that if $g : B \to A$ and $g' : B \to A$ are inverses of $f : A \to B$, then $g = g'$, i.e., for all $y \in B$, $g(y) = g'(y)$.

## fun.5 Composition of Functions

sfr:fun:cmp:
sec

We saw in section fun.4 that the inverse $f^{-1}$ of a bijection $f$ is itself a function. Another operation on functions is composition: we can define a new function by composing two functions, $f$ and $g$, i.e., by first applying $f$ and then $g$. Of course, this is only possible if the ranges and domains match, i.e., the range of $f$ must be a subset of the domain of $g$. explanation

A diagram might help to explain the idea of composition. In Figure fun.5, we depict two functions $f : A \to B$ and $g : B \to C$ and their composition $(g \circ f)$. The function $(g \circ f) : A \to C$ pairs each element of $A$ with an element of $C$. We specify which element of $C$ an element of $A$ is paired with as follows: given an input $x \in A$, first apply the function $f$ to $x$, which will output some $f(x) = y \in B$, then apply the function $g$ to $y$, which will output some $g(f(x)) = g(y) = z \in C$.

**Definition fun.18 (Composition).** Let $f : A \to B$ and $g : B \to C$ be functions. The *composition* of $f$ with $g$ is $g \circ f : A \to C$, where $(g \circ f)(x) = g(f(x))$.

**Example fun.19.** Consider the functions $f(x) = x + 1$, and $g(x) = 2x$. Since $(g \circ f)(x) = g(f(x))$, for each input $x$ you must first take its successor, then multiply the result by two. So their composition is given by $(g \circ f)(x) = 2(x+1)$.

**Problem fun.4.** Show that if $f : A \to B$ and $g : B \to C$ are both injective, then $g \circ f : A \to C$ is injective.

**Problem fun.5.** Show that if $f : A \to B$ and $g : B \to C$ are both surjective, then $g \circ f : A \to C$ is surjective.

**Problem fun.6.** Suppose $f : A \to B$ and $g : B \to C$. Show that the graph of $g \circ f$ is $R_f \mid R_g$.

## fun.6 Partial Functions

explanation It is sometimes useful to relax the definition of function so that it is not required that the output of the function is defined for all possible inputs. Such mappings are called *partial functions*. sfr:fun:par:sec

**Definition fun.20.** A *partial function* $f : A \nrightarrow B$ is a mapping which assigns to every element of $A$ at most one element of $B$. If $f$ assigns an element of $B$ to $x \in A$, we say $f(x)$ is *defined*, and otherwise *undefined*. If $f(x)$ is defined, we write $f(x) \downarrow$, otherwise $f(x) \uparrow$. The *domain* of a partial function $f$ is the subset of $A$ where it is defined, i.e., $\operatorname{dom}(f) = \{x \in A : f(x) \downarrow\}$.

**Example fun.21.** Every function $f : A \to B$ is also a partial function. Partial functions that are defined everywhere on $A$—i.e., what we so far have simply called a function—are also called *total* functions.

**Example fun.22.** The partial function $f : \mathbb{R} \nrightarrow \mathbb{R}$ given by $f(x) = 1/x$ is undefined for $x = 0$, and defined everywhere else.

**Problem fun.7.** Given $f : A \nrightarrow B$, define the partial function $g : B \nrightarrow A$ by: for any $y \in B$, if there is a unique $x \in A$ such that $f(x) = y$, then $g(y) = x$; otherwise $g(y) \uparrow$. Show that if $f$ is injective, then $g(f(x)) = x$ for all $x \in \operatorname{dom}(f)$, and $f(g(y)) = y$ for all $y \in \operatorname{ran}(f)$.

**Definition fun.23 (Graph of a partial function).** Let $f : A \nrightarrow B$ be a partial function. The *graph* of $f$ is the relation $R_f \subseteq A \times B$ defined by

$$R_f = \{\langle x, y \rangle : f(x) = y\}.$$

**Proposition fun.24.** *Suppose $R \subseteq A \times B$ has the property that whenever $Rxy$ and $Rxy'$ then $y = y'$. Then $R$ is the graph of the partial function $f : X \nrightarrow Y$ defined by: if there is a $y$ such that $Rxy$, then $f(x) = y$, otherwise $f(x) \uparrow$. If $R$ is also* serial, *i.e., for each $x \in X$ there is a $y \in Y$ such that $Rxy$, then $f$ is total.*

*Proof.* Suppose there is a $y$ such that $Rxy$. If there were another $y' \neq y$ such that $Rxy'$, the condition on $R$ would be violated. Hence, if there is a $y$ such that $Rxy$, that $y$ is unique, and so $f$ is well-defined. Obviously, $R_f = R$ and $f$ is total if $R$ is serial. $\square$

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