fun.1 Functions and Relations

A function which maps elements of $X$ to elements of $Y$ obviously defines a relation between $X$ and $Y$, namely the relation which holds between $x$ and $y$ iff $f(x) = y$. In fact, we might even—if we are interested in reducing the building blocks of mathematics for instance—identify the function $f$ with this relation, i.e., with a set of pairs. This then raises the question: which relations define functions in this way?

**Definition fun.1** (Graph of a function). Let $f : X \rightarrow Y$ be a partial function. The graph of $f$ is the relation $R_f \subseteq X \times Y$ defined by

$$R_f = \{ (x, y) : f(x) = y \}.$$

**Proposition fun.2.** Suppose $R \subseteq X \times Y$ has the property that whenever $Rxy$ and $Rxy'$ then $y = y'$. Then $R$ is the graph of the partial function $f : X \rightarrow Y$ defined by: if there is a $y$ such that $Rxy$, then $f(x) = y$, otherwise $f(x) \uparrow$. If $R$ is also serial, i.e., for each $x \in X$ there is a $y \in Y$ such that $Rxy$, then $f$ is total.

*Proof.* Suppose there is a $y$ such that $Rxy$. If there were another $y' \neq y$ such that $Rxy'$, the condition on $R$ would be violated. Hence, if there is a $y$ such that $Rxy$, that $y$ is unique, and so $f$ is well-defined. Obviously, $R_f = R$ and $f$ is total if $R$ is serial. \hfill $\Box$

**Problem fun.1.** Suppose $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. Show that the graph of $(g \circ f)$ is $R_f \mid R_g$.

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Bibliography