fun.1 Functions as Relations

sfr:fun:rel: sec A function which maps elements of A to elements of B obviously defines a relation between A and B, namely the relation which holds between x and y iff $f(x) = y$. In fact, we might even—if we are interested in reducing the building blocks of mathematics for instance—*identify* the function f with this relation, i.e., with a set of pairs. This then raises the question: which relations define functions in this way? explanation

Definition fun.1 (Graph of a function). Let $f : A \to B$ be a function. The *graph* of f is the relation $R_f \subseteq A \times B$ defined by

$$R_f = \{\langle x, y\rangle : f(x) = y\}.$$

The graph of a function is uniquely determined, by extensionality. Moreover, extensionality (on sets) will immediately vindicate the implicit principle of extensionality for functions, whereby if f and g share a domain and codomain then they are identical if they agree on all values. explanation

Similarly, if a relation is "functional", then it is the graph of a function.

sfr:fun:rel: prop:graph-function **Proposition fun.2.** *Let $R \subseteq A \times B$ be such that:*

1. *If Rxy and Rxz then $y = z$; and*
2. *for every $x \in A$ there is some $y \in B$ such that $\langle x, y\rangle \in R$.*

Then R is the graph of the function $f : A \to B$ defined by $f(x) = y$ iff Rxy.

Proof. Suppose there is a y such that Rxy. If there were another $z \neq y$ such that Rxz, the condition on R would be violated. Hence, if there is a y such that Rxy, this y is unique, and so f is well-defined. Obviously, $R_f = R$. □

Every function $f : A \to B$ has a graph, i.e., a relation on $A \times B$ defined by $f(x) = y$. On the other hand, every relation $R \subseteq A \times B$ with the properties given in Proposition fun.2 is the graph of a function $f : A \to B$. Because of this close connection between functions and their graphs, we can think of a function simply as its graph. In other words, functions can be identified with certain relations, i.e., with certain sets of tuples. We can now consider performing similar operations on functions as we performed on relations (see **??**). In particular: explanation

sfr:fun:rel: defn:funimage **Definition fun.3.** Let $f : A \to B$ be a function with $C \subseteq A$.

The *restriction* of f to C is the function $f{\restriction}_C : C \to B$ defined by $(f{\restriction}_C)(x) = f(x)$ for all $x \in C$. In other words, $f{\restriction}_C = \{\langle x, y\rangle \in R_f : x \in C\}$.

The *application* of f to C is $f[C] = \{f(x) : x \in C\}$. We also call this the *image* of C under f.

explanation

It follows from these definitions that $\mathrm{ran}(f) = f[\mathrm{dom}(f)]$, for any function f.

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Bibliography