fun.1 Composition of Functions

We have already seen that the inverse $f^{-1}$ of a bijective function $f$ is itself a function. It is also possible to compose functions $f$ and $g$ to define a new function by first applying $f$ and then $g$. Of course, this is only possible if the ranges and domains match, i.e., the range of $f$ must be a subset of the domain of $g$.

Definition fun.1 (Composition). Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$. The composition of $f$ with $g$ is the function $(g \circ f): X \rightarrow Z$, where $(g \circ f)(x) = g(f(x))$.

The function $(g \circ f): X \rightarrow Z$ pairs each member of $X$ with a member of $Z$. We specify which member of $Z$ a member of $X$ is paired with as follows—given an input $x \in X$, first apply the function $f$ to $x$, which will output some $y \in Y$. Then apply the function $g$ to $y$, which will output some $z \in Z$.

Example fun.2. Consider the functions $f(x) = x + 1$, and $g(x) = 2x$. What function do you get when you compose these two? $(g \circ f)(x) = g(f(x))$. So that means for every natural number you give this function, you first add one, and then you multiply the result by two. So their composition is $(g \circ f)(x) = 2(x+1)$.

Problem fun.1. Show that if $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are both injective, then $g \circ f: X \rightarrow Z$ is injective.

Problem fun.2. Show that if $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are both surjective, then $g \circ f: X \rightarrow Z$ is surjective.

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Bibliography