arith.1 The Real Line

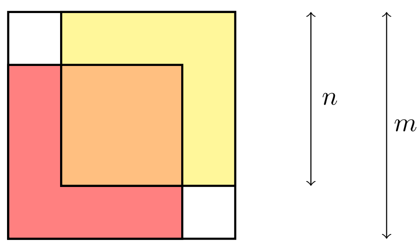
sfr:arith:real:sec The next step is to show how to construct the reals from the rationals. Before that, we need to understand what is *distinctive* about the reals.

The reals behave very much like the rationals. (Technically, both are examples of *ordered fields*; for the definition of this, see **??**.) Now, if you worked through the exercises to **??**, you will know that there are strictly more reals than rationals, i.e., that $\mathbb{Q} \prec \mathbb{R}$. This was first proved by Cantor. But it's been known for about two and a half millennia that there are irrational numbers, i.e., reals which are not rational. Indeed:

sfr:arith:real:root2irrational **Theorem arith.1.** *$\sqrt{2}$ is not rational, i.e., $\sqrt{2} \notin \mathbb{Q}$*

Proof. Suppose, for reductio, that $\sqrt{2}$ is rational. So $\sqrt{2} = m/n$ for some natural numbers m and n. Indeed, we can choose m and n so that the fraction cannot be reduced any further. Re-organising, $m^2 = 2n^2$. From here, we can complete the proof in two ways:

First, geometrically (following Tennenbaum).[1] Consider these squares:



□

Since $m^2 = 2n^2$, the region where the two squares of side n overlap has the same area as the region which neither of the two squares cover; i.e., the area of the orange square equals the sum of the area of the two unshaded squares. So where the orange square has side p, and each unshaded square has side q, $p^2 = 2q^2$. But now $\sqrt{2} = p/q$, with $p < m$ and $q < n$ and $p, q \in \mathbb{N}$. This contradicts the fact that m and n were chosen to be as small as possible.

Second, formally. Since $m^2 = 2n^2$, it follows that m is even. (It is easy to show that, if x is odd, then x^2 is odd.) So $m = 2r$, for some $r \in \mathbb{N}$. Rearranging, $2r^2 = n^2$, so n is also even. So both m and n are even, and hence the fraction m/n *can* be reduced further. Contradiction!

In passing, this diagrammatic proof allows us to revisit the material from **??**. Tennenbaum (1927–2006) was a thoroughly modern mathematician; but the proof is undeniably lovely, completely rigorous, and appeals to geometric intuition!

In any case: the reals are "more expansive" than the rationals. In some sense, there are "gaps" in the rationals, and these are filled by the reals. Weierstrass realised that this describes a single property of the real numbers, which

[1] This proof is reported by Conway (2006).

distinguishes them from the rationals, namely the Completeness Property: *Every non-empty set of real numbers with an upper bound has a least upper bound.*

It is easy to see that the rationals do not have the Completeness Property. For example, consider the set of rationals less than $\sqrt{2}$, i.e.:

$$\{p \in \mathbb{Q} : p^2 < 2 \text{ or } p < 0\}$$

This has an upper bound in the rationals; its elements are all smaller than 3, for example. But what is its least upper bound? We want to say '$\sqrt{2}$'; but we have just seen that $\sqrt{2}$ is *not* rational. And there is no *least* rational number greater than $\sqrt{2}$. So the set has an upper bound but no least upper bound. Hence the rationals lack the Completeness Property.

By contrast, the continuum "morally ought" to have the Completeness Property. We do not just want $\sqrt{2}$ to be a real number; we want to fill all the "gaps" in the rational line. Indeed, we want the continuum itself to have no "gaps" in it. That is just what we will get via Completeness.

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Bibliography

Conway, John. 2006. The power of mathematics. In *Power*, eds. Alan Blackwell and David MacKay, Darwin College Lectures. Cambridge: Cambridge University Press. URL `http://www.cs.toronto.edu/~mackay/conway.pdf`.