arith.1 From $\mathbb{Z}$ to $\mathbb{Q}$

sfr:arith:rat:sec We just saw how to construct the integers from the natural numbers, using some naïve set theory. We shall now see how to construct the rationals from the integers in a very similar way. Our initial realisations are:

1. Every rational can be written in the form ${}^{i}/_{j}$, where both i and j are integers but j is non-zero.
2. The information encoded in an expression ${}^{i}/_{j}$ can equally be encoded in an ordered pair $\langle i, j\rangle$.

The obvious approach would be to think of the rationals *as* ordered pairs drawn from $\mathbb{Z} \times (\mathbb{Z} \setminus \{0_{\mathbb{Z}}\})$. As before, though, that would be a bit too naïve, since we want ${}^{3}/_{2} = {}^{6}/_{4}$, but $\langle 3, 2\rangle \neq \langle 6, 4\rangle$. More generally, we will want the following:

$$ {}^{a}/_{b} = {}^{c}/_{d} \text{ iff } a \times d = b \times c $$

To get this, we define an equivalence relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0_{\mathbb{Z}}\})$ thus:

$$ \langle a, b\rangle \backsim \langle c, d\rangle \text{ iff } a \times d = b \times c $$

We must check that this is an equivalence relation. This is very much like the case of $\sim$, and we will leave it as an exercise.

Problem arith.1. Show that $\backsim$ is an equivalence relation.

But it allows us to say:

Definition arith.1. The rationals are the equivalence classes, under $\backsim$, of pairs of integers (whose second element is non-zero). That is, $\mathbb{Q} = (\mathbb{Z} \times (\mathbb{Z} \setminus \{0_{\mathbb{Z}}\}))/_{\backsim}$.

As with the integers, we also want to define some basic operations. Where $[i, j]_{\backsim}$ is the equivalence class under $\backsim$ with $\langle i, j\rangle$ as an element, we say:

$$ \begin{aligned} [a, b]_{\backsim} + [c, d]_{\backsim} &= [ad + bc, bd]_{\backsim} \\ [a, b]_{\backsim} \times [c, d]_{\backsim} &= [ac, bd]_{\backsim}. \end{aligned} $$

To define $r \leq s$ on these rationals, we use the fact that $r \leq s$ iff $s - r$ is not negative, i.e., $r - s$ can be written as ${}^{i}/_{j}$ with i non-negative and j positive:

$$ [a, b]_{\backsim} \leq [c, d]_{\backsim} \text{ iff } [c, d]_{\backsim} - [a, b]_{\backsim} = [i_{\mathbb{Z}}, j_{\mathbb{Z}}]_{\backsim} $$

for some $i \in \mathbb{N}$ and $0 \neq j \in \mathbb{N}$.

We then need to check that these definitions behave as they *ought* to; and we relegate this to **??**. But they indeed do! Finally, we want some way to treat integers *as* rationals; so for each $i \in \mathbb{Z}$, we stipulate that $i_{\mathbb{Q}} = [i, 1_{\mathbb{Z}}]_{\backsim}$. Again, we check that all of this behaves correctly in **??**.

Problem arith.2. Show that $(i + j)_{\mathbb{Q}} = i_{\mathbb{Q}} + j_{\mathbb{Q}}$ and $(i \times j)_{\mathbb{Q}} = i_{\mathbb{Q}} \times j_{\mathbb{Q}}$ and $i \leq j \leftrightarrow i_{\mathbb{Q}} \leq j_{\mathbb{Q}}$, for any $i, j \in \mathbb{Z}$.

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Bibliography