

## arith.1 From $\mathbb{Z}$ to $\mathbb{Q}$

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We just saw how to construct the integers from the natural numbers, using some naïve set theory. We shall now see how to construct the rationals from the integers in a very similar way. Our initial realisations are:

1. Every rational can be written in the form  $i/j$ , where both  $i$  and  $j$  are integers but  $j$  is non-zero.
2. The information encoded in an expression  $i/j$  can equally be encoded in an ordered pair  $\langle i, j \rangle$ .

The obvious approach would be to think of the rationals *as* ordered pairs drawn from  $\mathbb{Z} \times (\mathbb{Z} \setminus \{0_{\mathbb{Z}}\})$ . As before, though, that would be a bit too naïve, since we want  $3/2 = 6/4$ , but  $\langle 3, 2 \rangle \neq \langle 6, 4 \rangle$ . More generally, we will want the following:

$$a/b = c/d \text{ iff } a \times d = b \times c$$

To get this, we define an equivalence relation on  $\mathbb{Z} \times (\mathbb{Z} \setminus \{0_{\mathbb{Z}}\})$  thus:

$$\langle a, b \rangle \sim \langle c, d \rangle \text{ iff } a \times d = b \times c$$

We must check that this is an equivalence relation. This is very much like the case of  $\sim$ , and we will leave it as an exercise.

**Problem arith.1.** Show that  $\sim$  is an equivalence relation.

But this allows us to say:

**Definition arith.1.** The rationals are the equivalence classes, under  $\sim$ , of pairs of integers (whose second element is non-zero):  $\mathbb{Q} = (\mathbb{Z} \times (\mathbb{Z} \setminus \{0_{\mathbb{Z}}\})) / \sim$ .

As with the integers, we also want to define some basic operations. Where  $[i, j]_{\sim}$  is the equivalence class under  $\sim$  with  $\langle i, j \rangle$  as **an element**, we say:

$$[a, b]_{\sim} + [c, d]_{\sim} = [ad + bc, bd]_{\sim}$$

$$[a, b]_{\sim} \times [c, d]_{\sim} = [ac, bd]_{\sim}$$

$$[a, b]_{\sim} \leq [c, d]_{\sim} \text{ iff } ad \leq bc$$

We then need to check that these definitions behave as they *ought* to; and we relegate this to ???. But they indeed do! Finally, we want some way to treat integers *as* rationals; so for each  $i \in \mathbb{Z}$ , we stipulate that  $i_{\mathbb{Q}} = [i, 1_{\mathbb{Z}}]_{\sim}$ . Again, we check that all of this behaves correctly in ???.

**Problem arith.2.** Show that  $(i + j)_{\mathbb{Q}} = i_{\mathbb{Q}} + j_{\mathbb{Q}}$  and  $(i \times j)_{\mathbb{Q}} = i_{\mathbb{Q}} \times j_{\mathbb{Q}}$  and  $i \leq j \leftrightarrow i_{\mathbb{Q}} \leq j_{\mathbb{Q}}$ , for any  $i, j \in \mathbb{Z}$ .

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**Bibliography**