We just saw how to construct the integers from the natural numbers, using some naïve set theory. We shall now see how to construct the rationals from the integers in a very similar way. Our initial realisations are:

1. Every rational can be written in the form \( \frac{i}{j} \), where both \( i \) and \( j \) are integers but \( j \) is non-zero.

2. The information encoded in an expression \( \frac{i}{j} \) can equally be encoded in an ordered pair \( \langle i, j \rangle \).

The obvious approach would be to think of the rationals as ordered pairs drawn from \( \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \). As before, though, that would be a bit too naïve, since we want \( \frac{3}{2} = \frac{6}{4} \), but \( \langle 3, 2 \rangle \neq \langle 6, 4 \rangle \). More generally, we will want the following:

\[
\frac{a}{b} = \frac{c}{d} \text{ iff } a \times d = b \times c
\]

To get this, we define an equivalence relation on \( \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \) thus:

\[
\langle a, b \rangle \sim \langle c, d \rangle \text{ iff } a \times d = b \times c
\]

We must check that this is an equivalence relation. This is very much like the case of \( \sim \), and we will leave it as an exercise.

**Problem arith.1.** Show that \( \sim \) is an equivalence relation.

But it allows us to say:

**Definition arith.1.** The rationals are the equivalence classes, under \( \sim \), of pairs of integers (whose second element is non-zero). That is, \( \mathbb{Q} = (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}))/\sim \).

As with the integers, we also want to define some basic operations. Where \( [i, j]_\sim \) is the equivalence class under \( \sim \) with \( \langle i, j \rangle \) as an element, we say:

\[
\begin{align*}
[a, b]_\sim + [c, d]_\sim &= [ad + bc, bd]_\sim \\
[a, b]_\sim \times [c, d]_\sim &= [ac, bd]_\sim
\end{align*}
\]

To define \( r \leq s \) on these rationals, we use the fact that \( r \leq s \) iff \( s - r \) is not negative, i.e., \( r - s \) can be written as \( \frac{i}{j} \) with \( i \) non-negative and \( j \) positive:

\[
[a, b]_\sim \leq [c, d]_\sim \text{ iff } [c, d]_\sim - [a, b]_\sim = [iZ, jZ]_\sim
\]

for some \( i \in \mathbb{N} \) and \( 0 \neq j \in \mathbb{N} \).

We then need to check that these definitions behave as they ought to; and we relegate this to ???. But they indeed do! Finally, we want some way to treat integers as rationals; so for each \( i \in \mathbb{Z} \), we stipulate that \( iQ = [i, 1Z]_\sim \). Again, we check that all of this behaves correctly in ???.

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Problem arith.2. Show that \((i + j)Q = iQ + jQ\) and \((i \times j)Q = iQ \times jQ\) and \(i \leq j \leftrightarrow iQ \leq jQ\), for any \(i, j \in \mathbb{Z}\).