We just saw how to construct the integers from the natural numbers, using some na"ıve set theory. We shall now see how to construct the rationals from the integers in a very similar way. Our initial realisations are:

1. Every rational can be written in the form $i/j$, where both $i$ and $j$ are integers but $j$ is non-zero.

2. The information encoded in an expression $i/j$ can equally be encoded in an ordered pair $\langle i, j \rangle$.

The obvious approach would be to think of the rationals as ordered pairs drawn from $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. As before, though, that would be a bit too na"ıve, since we want $3/2 = 6/4$, but $\langle 3, 2 \rangle \neq \langle 6, 4 \rangle$. More generally, we will want the following:

$$a/b = c/d \iff a \times d = b \times c$$

To get this, we define an equivalence relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ thus:

$$\langle a, b \rangle \sim \langle c, d \rangle \iff a \times d = b \times c$$

We must check that this is an equivalence relation. This is very much like the case of $\sim$, and we will leave it as an exercise.

**Problem arith.1.** Show that $\sim$ is an equivalence relation.

But it allows us to say:

**Definition arith.1.** The rationals are the equivalence classes, under $\sim$, of pairs of integers (whose second element is non-zero). That is, $\mathbb{Q} = (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}))/\sim$.

As with the integers, we also want to define some basic operations. Where $[i, j]_\sim$ is the equivalence class under $\sim$ with $\langle i, j \rangle$ as an element, we say:

$$(a, b) \sim (c, d) \iff a \times d = b \times c$$

To define $r \leq s$ on these rationals, we use the fact that $r \leq s$ iff $s - r$ is not negative, i.e., $r - s$ can be written as $i/j$ with $i$ non-negative and $j$ positive:

$$[a, b]_\sim \leq [c, d]_\sim \iff [c, d]_\sim - [a, b]_\sim = [i, j]_\sim$$

for some $i \in \mathbb{N}$ and $0 \neq j \in \mathbb{N}$.

We then need to check that these definitions behave as they ought to; and we relegate this to ???. But they indeed do! Finally, we want some way to treat integers as rationals; so for each $i \in \mathbb{Z}$, we stipulate that $i_\mathbb{Q} = [i, 1]_\sim$. Again, we check that all of this behaves correctly in ??.
Problem arith.2. Show that $(i + j)_Q = i_Q + j_Q$ and $(i \times j)_Q = i_Q \times j_Q$ and $i \leq j \leftrightarrow i_Q \leq j_Q$, for any $i, j \in \mathbb{Z}$.