arith.1 From $\mathbb{Q}$ to $\mathbb{R}$

sfr:arith:cuts: sec

In essence, the Completeness Property shows that any point α of the real line divides that line into two halves perfectly: those for which α is the least upper bound, and those for which α is the greatest lower bound. To *construct* the real numbers from the rational numbers, Dedekind suggested that we simply think of the reals as the *cuts* that partition the rationals. That is, we identify $\sqrt{2}$ with the *cut* which separates the rationals $< \sqrt{2}$ from the rationals $> \sqrt{2}$.

Let's tidy this up. If we cut the rational numbers into two halves, we can uniquely identify the partition we made just by considering its *bottom* half. So, getting precise, we offer the following definition:

Definition arith.1 (Cut). A *cut* α is any non-empty proper initial segment of the rationals with no greatest element. That is, α is a cut iff:

1. *non-empty, proper*: $\emptyset \neq \alpha \subsetneq \mathbb{Q}$
2. *initial*: for all $p, q \in \mathbb{Q}$: if $p < q \in \alpha$ then $p \in \alpha$
3. *no maximum*: for all $p \in \alpha$ there is a $q \in \alpha$ such that $p < q$

Then $\mathbb{R}$ is the set of cuts.

So now we can say that $\sqrt{2} = \{p \in \mathbb{Q} : p^2 < 2 \text{ or } p < 0\}$. Of course, we need to check that this *is* a cut, but we relegate that to **??**.

As before, having defined some entities, we next need to define basic functions and relations upon them. We begin with an easy one:

$$\alpha \leq \beta \text{ iff } \alpha \subseteq \beta$$

This definition of an order allows to *state* the central result, that the set of cuts has the Completeness Property. Spelled out fully, the statement has this shape. If S is a non-empty set of cuts with an upper bound, then S has a least upper bound. In more detail: there is a cut, λ, which is an upper bound for S, i.e. $(\forall \alpha \in S)\alpha \subseteq \lambda$, and λ is the least such cut, i.e. $(\forall \beta \in \mathbb{R})((\forall \alpha \in S)\alpha \subseteq \beta \rightarrow \lambda \subseteq \beta)$. Now here is the proof of the result:

sfr:arith:cuts: realcompleteness

Theorem arith.2. *The set of cuts has the Completeness Property.*

Proof. Let S be any non-empty set of cuts with an upper bound. Let $\lambda = \bigcup S$. We first claim that λ is a cut:

1. Since S has an upper bound, at least one cut is in S, so $\emptyset \neq \lambda$. Since S is a set of cuts, $\lambda \subseteq \mathbb{Q}$. Since S has an upper bound, some $p \in \mathbb{Q}$ is absent from every cut $\alpha \in S$. So $p \notin \lambda$, and hence $\lambda \subsetneq \mathbb{Q}$.
2. Suppose $p < q \in \lambda$. So there is some $\alpha \in S$ such that $q \in \alpha$. Since α is a cut, $p \in \alpha$. So $p \in \lambda$.

3. Suppose $p \in \lambda$. So there is some $\alpha \in S$ such that $p \in \alpha$. Since α is a cut, there is some $q \in \alpha$ such that $p < q$. So $q \in \lambda$.

This proves the claim. Moreover, clearly $(\forall \alpha \in S)\alpha \subseteq \bigcup S = \lambda$, i.e. λ is an upper bound on S. So now suppose $\beta \in \mathbb{R}$ is also an upper bound, i.e. $(\forall \alpha \in S)\alpha \subseteq \beta$. For any $p \in \mathbb{Q}$, if $p \in \lambda$, then there is $\alpha \in S$ such that $p \in \alpha$, so that $p \in \beta$. Generalizing, $\lambda \subseteq \beta$. So λ is the *least* upper bound on S. □

So we have a bunch of entities which satisfy the Completeness Property. And one way to put this is: there are no "gaps" in our cuts. (So: taking further "cuts" of reals, rather than rationals, would yield no interesting new objects.)

Next, we must define some operations on the reals. We start by embedding the rationals into the reals by stipulating that $p_{\mathbb{R}} = \{q \in \mathbb{Q} : q < p\}$ for each $p \in \mathbb{Q}$. We then define:

$$\begin{aligned} \alpha + \beta &= \{p + q : p \in \alpha \land q \in \beta\} \\ \alpha \times \beta &= \{p \times q : 0 \le p \in \alpha \land 0 \le q \in \beta\} \cup 0^{\mathbb{R}} && \text{if } \alpha, \beta \ge 0_{\mathbb{R}} \end{aligned}$$

To handle the other multiplication cases, first let:

$$-\alpha = \{p - q : p < 0 \land q \notin \alpha\}$$

and then stipulate:

$$\alpha \times \beta = \begin{cases} -\alpha \times -\beta & \text{if } \alpha < 0_{\mathbb{R}} \text{ and } \beta < 0_{\mathbb{R}} \\ -(-\alpha \times \beta) & \text{if } \alpha < 0_{\mathbb{R}} \text{ and } \beta > 0_{\mathbb{R}} \\ -(\alpha \times -\beta) & \text{if } \alpha > 0_{\mathbb{R}} \text{ and } \beta < 0_{\mathbb{R}} \end{cases}$$

We then need to check that each of these definitions always yields a cut. And finally, we need to go through an easy (but long-winded) demonstration that the cuts, so defined, behave exactly as they should. But we relegate all of this to **??**.

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Bibliography