udf Arithmetization

The material in this chapter presents the construction of the number systems in naïve set theory. It is taken from Tim Button's Open Set Theory text.

arith.1 From $\mathbb{N}$ to $\mathbb{Z}$

sfr:arith:int:sec

Here are two basic realisations:

1. Every integer can be written in the form $n - m$, with $n, m \in \mathbb{N}$.
2. The information encoded in an expression $n - m$ can equally be encoded by an ordered pair $\langle n, m \rangle$.

We already know that the ordered pairs of natural numbers are the elements of $\mathbb{N}^2$. And we are assuming that we understand $\mathbb{N}$. So here is a naïve suggestion, based on the two realisations we have had: *let's treat integers as ordered pairs of natural numbers.*

In fact, this suggestion is too naïve. Obviously we want it to be the case that $0 - 2 = 4 - 6$. But evidently $\langle 0, 2 \rangle \neq \langle 4, 6 \rangle$. So we cannot simply say that $\mathbb{N}^2$ is the set of integers.

Generalising from the preceding problem, what we want is the following:

$$a - b = c - d \text{ iff } a + d = c + b$$

(It should be obvious that this is how integers are *meant* to behave: just add b and d to both sides.) And the easy way to guarantee this behaviour is just to define an equivalence relation between ordered pairs, $\sim$, as follows:

$$\langle a, b \rangle \sim \langle c, d \rangle \text{ iff } a + d = c + b$$

We now have to show that this is an equivalence relation.

Proposition arith.1. *$\sim$ is an equivalence relation.*

Proof. We must show that $\sim$ is reflexive, symmetric, and transitive.

Reflexivity: Evidently $\langle a, b \rangle \sim \langle a, b \rangle$, since $a + b = b + a$.

Symmetry: Suppose $\langle a, b \rangle \sim \langle c, d \rangle$, so $a + d = c + b$. Then $c + b = a + d$, so that $\langle c, d \rangle \sim \langle a, b \rangle$.

Transitivity: Suppose $\langle a, b \rangle \sim \langle c, d \rangle \sim \langle m, n \rangle$. So $a + d = c + b$ and $c + n = m + d$. So $a + d + c + n = c + b + m + d$, and so $a + n = m + b$. Hence $\langle a, b \rangle \sim \langle m, n \rangle$. □

Now we can use this equivalence relation to take equivalence classes:

Definition arith.2. The integers are the equivalence classes, under $\sim$, of ordered pairs of natural numbers; that is, $\mathbb{Z} = \mathbb{N}^2/_{\sim}$.

Now, one might have plenty of different *philosophical* reactions to this stipulative definition. Before we consider those reactions, though, it is worth continuing with some of the technicalities.

Having said what the integers are, we shall need to define basic functions and relations on them. Let's write $[m, n]_\sim$ for the equivalence class under $\sim$ with $\langle m, n\rangle$ as an element.[1] That is:

$$[m, n]_\sim = \{\langle a, b\rangle \in \mathbb{N}^2 : \langle a, b\rangle \sim \langle m, n\rangle\}$$

So now we offer some definitions:

$$\begin{aligned}
[a, b]_\sim + [c, d]_\sim &= [a + c, b + d]_\sim \\
[a, b]_\sim \times [c, d]_\sim &= [ac + bd, ad + bc]_\sim \\
[a, b]_\sim \leq [c, d]_\sim &\text{ iff } a + d \leq b + c
\end{aligned}$$

(As is common, I'm using 'ab' stand for '$(a \times b)$', just to make the axioms easier to read.) Now, we need to make sure that these definitions behave as they *ought* to. Spelling out what this means, and checking it through, is rather laborious; we relegate the details to section arith.6. But the short point is: everything works!

One final thing remains. We have constructed the integers using natural numbers. But this will mean that the natural numbers *are not themselves integers*. We will return to the philosophical significance of this in section arith.5. On a purely technical front, though, we will need some way to be able to treat natural numbers *as* integers. The idea is quite easy: for each $n \in \mathbb{N}$, we just stipulate that $n_\mathbb{Z} = [n, 0]_\sim$. We need to confirm that this definition is well-behaved, i.e., that for any $m, n \in \mathbb{N}$

$$\begin{aligned}
(m + n)_\mathbb{Z} &= m_\mathbb{Z} + n_\mathbb{Z} \\
(m \times n)_\mathbb{Z} &= m_\mathbb{Z} \times n_\mathbb{Z} \\
m \leq n &\leftrightarrow m_\mathbb{Z} \leq n_\mathbb{Z}
\end{aligned}$$

But this is all pretty straightforward. For example, to show that the second of these obtains, we can simply help ourselves to the behaviour of the natural numbers and reason as follows:

$$\begin{aligned}
(m \times n)_\mathbb{Z} &= [m \times n, 0]_\sim \\
&= [m \times n + 0 \times 0, m \times 0 + 0 \times n]_\sim \\
&= [m, 0]_\sim \times [n, 0]_\sim \\
&= m_\mathbb{Z} \times n_\mathbb{Z}
\end{aligned}$$

We leave it as an exercise to confirm that the other two conditions hold.

Problem arith.1. Show that $(m + n)_\mathbb{Z} = m_\mathbb{Z} + n_\mathbb{Z}$ and $m \leq n \leftrightarrow m_\mathbb{Z} \leq n_\mathbb{Z}$, for any $m, n \in \mathbb{N}$.

[1] Note: using the notation introduced in **??**, we would have written $[\langle m, n\rangle]_\sim$ for the same thing. But that's just a bit harder to read.

arith.2 From $\mathbb{Z}$ to $\mathbb{Q}$

sfr:arith:rat:sec

We just saw how to construct the integers from the natural numbers, using some naïve set theory. We shall now see how to construct the rationals from the integers in a very similar way. Our initial realisations are:

1. Every rational can be written in the form $^i/_j$, where both i and j are integers but j is non-zero.
2. The information encoded in an expression $^i/_j$ can equally be encoded in an ordered pair $\langle i, j\rangle$.

The obvious approach would be to think of the rationals *as* ordered pairs drawn from $\mathbb{Z} \times (\mathbb{Z} \setminus \{0_{\mathbb{Z}}\})$. As before, though, that would be a bit too naïve, since we want $^3/_2 = {}^6/_4$, but $\langle 3, 2\rangle \neq \langle 6, 4\rangle$. More generally, we will want the following:

$$^a/_b = {}^c/_d \text{ iff } a \times d = b \times c$$

To get this, we define an equivalence relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0_{\mathbb{Z}}\})$ thus:

$$\langle a, b\rangle \backsim \langle c, d\rangle \text{ iff } a \times d = b \times c$$

We must check that this is an equivalence relation. This is very much like the case of $\sim$, and we will leave it as an exercise.

Problem arith.2. Show that $\backsim$ is an equivalence relation.

But it allows us to say:

Definition arith.3. The rationals are the equivalence classes, under $\backsim$, of pairs of integers (whose second element is non-zero). That is, $\mathbb{Q} = (\mathbb{Z} \times (\mathbb{Z} \setminus \{0_{\mathbb{Z}}\}))/_{\backsim}$.

As with the integers, we also want to define some basic operations. Where $[i, j]_{\backsim}$ is the equivalence class under $\backsim$ with $\langle i, j\rangle$ as an element, we say:

$$[a, b]_{\backsim} + [c, d]_{\backsim} = [ad + bc, bd]_{\backsim}$$
$$[a, b]_{\backsim} \times [c, d]_{\backsim} = [ac, bd]_{\backsim}.$$

To define $r \leq s$ on these rationals, we use the fact that $r \leq s$ iff $s - r$ is not negative, i.e., $r - s$ can be written as $^i/_j$ with i non-negative and j positive:

$$[a, b]_{\backsim} \leq [c, d]_{\backsim} \text{ iff } [c, d]_{\backsim} - [a, b]_{\backsim} = [i_{\mathbb{Z}}, j_{\mathbb{Z}}]_{\backsim}$$

for some $i \in \mathbb{N}$ and $0 \neq j \in \mathbb{N}$.

We then need to check that these definitions behave as they *ought* to; and we relegate this to section arith.6. But they indeed do! Finally, we want some way to treat integers *as* rationals; so for each $i \in \mathbb{Z}$, we stipulate that $i_{\mathbb{Q}} = [i, 1_{\mathbb{Z}}]_{\backsim}$. Again, we check that all of this behaves correctly in section arith.6.

Problem arith.3. Show that $(i + j)_{\mathbb{Q}} = i_{\mathbb{Q}} + j_{\mathbb{Q}}$ and $(i \times j)_{\mathbb{Q}} = i_{\mathbb{Q}} \times j_{\mathbb{Q}}$ and $i \leq j \leftrightarrow i_{\mathbb{Q}} \leq j_{\mathbb{Q}}$, for any $i, j \in \mathbb{Z}$.

arith.3 The Real Line

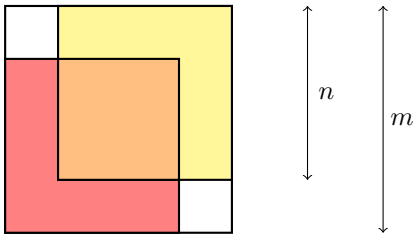
The next step is to show how to construct the reals from the rationals. Before that, we need to understand what is *distinctive* about the reals. sfr:arith:real:sec

The reals behave very much like the rationals. (Technically, both are examples of *ordered fields*; for the definition of this, see Definition arith.9.) Now, if you worked through the exercises to **??**, you will know that there are strictly more reals than rationals, i.e., that $\mathbb{Q} \prec \mathbb{R}$. This was first proved by Cantor. But it's been known for about two and a half millennia that there are irrational numbers, i.e., reals which are not rational. Indeed:

Theorem arith.4. *$\sqrt{2}$ is not rational, i.e., $\sqrt{2} \notin \mathbb{Q}$* sfr:arith:real:root2irrational

Proof. Suppose, for reductio, that $\sqrt{2}$ is rational. So $\sqrt{2} = {}^{m}/_{n}$ for some natural numbers m and n. Indeed, we can choose m and n so that the fraction cannot be reduced any further. Re-organising, $m^2 = 2n^2$. From here, we can complete the proof in two ways:

First, geometrically (following Tennenbaum).[2] Consider these squares:



□

Since $m^2 = 2n^2$, the region where the two squares of side n overlap has the same area as the region which neither of the two squares cover; i.e., the area of the orange square equals the sum of the area of the two unshaded squares. So where the orange square has side p, and each unshaded square has side q, $p^2 = 2q^2$. But now $\sqrt{2} = {}^{p}/_{q}$, with $p < m$ and $q < n$ and $p, q \in \mathbb{N}$. This contradicts the fact that m and n were chosen to be as small as possible.

Second, formally. Since $m^2 = 2n^2$, it follows that m is even. (It is easy to show that, if x is odd, then x^2 is odd.) So $m = 2r$, for some $r \in \mathbb{N}$. Rearranging, $2r^2 = n^2$, so n is also even. So both m and n are even, and hence the fraction ${}^{m}/_{n}$ *can* be reduced further. Contradiction!

In passing, this diagrammatic proof allows us to revisit the material from **??**. Tennenbaum (1927–2006) was a thoroughly modern mathematician; but the proof is undeniably lovely, completely rigorous, and appeals to geometric intuition!

In any case: the reals are "more expansive" than the rationals. In some sense, there are "gaps" in the rationals, and these are filled by the reals. Weierstrass realised that this describes a single property of the real numbers, which

[2] This proof is reported by Conway (2006).

distinguishes them from the rationals, namely the Completeness Property: *Every non-empty set of real numbers with an upper bound has a least upper bound.*

It is easy to see that the rationals do not have the Completeness Property. For example, consider the set of rationals less than $\sqrt{2}$, i.e.:

$$\{p \in \mathbb{Q} : p^2 < 2 \text{ or } p < 0\}$$

This has an upper bound in the rationals; its elements are all smaller than 3, for example. But what is its least upper bound? We want to say '$\sqrt{2}$'; but we have just seen that $\sqrt{2}$ is *not* rational. And there is no *least* rational number greater than $\sqrt{2}$. So the set has an upper bound but no least upper bound. Hence the rationals lack the Completeness Property.

By contrast, the continuum "morally ought" to have the Completeness Property. We do not just want $\sqrt{2}$ to be a real number; we want to fill all the "gaps" in the rational line. Indeed, we want the continuum itself to have no "gaps" in it. That is just what we will get via Completeness.

arith.4 From $\mathbb{Q}$ to $\mathbb{R}$

sfr:arith:cuts:sec

In essence, the Completeness Property shows that any point α of the real line divides that line into two halves perfectly: those for which α is the least upper bound, and those for which α is the greatest lower bound. To *construct* the real numbers from the rational numbers, Dedekind suggested that we simply think of the reals as the *cuts* that partition the rationals. That is, we identify $\sqrt{2}$ with the *cut* which separates the rationals $< \sqrt{2}$ from the rationals $> \sqrt{2}$.

Let's tidy this up. If we cut the rational numbers into two halves, we can uniquely identify the partition we made just by considering its *bottom* half. So, getting precise, we offer the following definition:

Definition arith.5 (Cut). A *cut* α is any non-empty proper initial segment of the rationals with no greatest element. That is, α is a cut iff:

1. *non-empty, proper*: $\emptyset \neq \alpha \subsetneq \mathbb{Q}$
2. *initial*: for all $p, q \in \mathbb{Q}$: if $p < q \in \alpha$ then $p \in \alpha$
3. *no maximum*: for all $p \in \alpha$ there is a $q \in \alpha$ such that $p < q$

Then $\mathbb{R}$ is the set of cuts.

So now we can say that $\sqrt{2} = \{p \in \mathbb{Q} : p^2 < 2 \text{ or } p < 0\}$. Of course, we need to check that this *is* a cut, but we relegate that to section arith.6.

As before, having defined some entities, we next need to define basic functions and relations upon them. We begin with an easy one:

$$\alpha \leq \beta \text{ iff } \alpha \subseteq \beta$$

This definition of an order allows to *state* the central result, that the set of cuts has the Completeness Property. Spelled out fully, the statement has this

shape. If S is a non-empty set of cuts with an upper bound, then S has a least upper bound. In more detail: there is a cut, λ, which is an upper bound for S, i.e. $(\forall \alpha \in S)\alpha \subseteq \lambda$, and λ is the least such cut, i.e. $(\forall \beta \in \mathbb{R})((\forall \alpha \in S)\alpha \subseteq \beta \to \lambda \subseteq \beta)$. Now here is the proof of the result:

Theorem arith.6. *The set of cuts has the Completeness Property.*

sfr:arith:cuts: realcompleteness

Proof. Let S be any non-empty set of cuts with an upper bound. Let $\lambda = \bigcup S$. We first claim that λ is a cut:

1. Since S has an upper bound, at least one cut is in S, so $\emptyset \neq \lambda$. Since S is a set of cuts, $\lambda \subseteq \mathbb{Q}$. Since S has an upper bound, some $p \in \mathbb{Q}$ is absent from every cut $\alpha \in S$. So $p \notin \lambda$, and hence $\lambda \subsetneq \mathbb{Q}$.

2. Suppose $p < q \in \lambda$. So there is some $\alpha \in S$ such that $q \in \alpha$. Since α is a cut, $p \in \alpha$. So $p \in \lambda$.

3. Suppose $p \in \lambda$. So there is some $\alpha \in S$ such that $p \in \alpha$. Since α is a cut, there is some $q \in \alpha$ such that $p < q$. So $q \in \lambda$.

This proves the claim. Moreover, clearly $(\forall \alpha \in S)\alpha \subseteq \bigcup S = \lambda$, i.e. λ is an upper bound on S. So now suppose $\beta \in \mathbb{R}$ is also an upper bound, i.e. $(\forall \alpha \in S)\alpha \subseteq \beta$. For any $p \in \mathbb{Q}$, if $p \in \lambda$, then there is $\alpha \in S$ such that $p \in \alpha$, so that $p \in \beta$. Generalizing, $\lambda \subseteq \beta$. So λ is the *least* upper bound on S. □

So we have a bunch of entities which satisfy the Completeness Property. And one way to put this is: there are no "gaps" in our cuts. (So: taking further "cuts" of reals, rather than rationals, would yield no interesting new objects.)

Next, we must define some operations on the reals. We start by embedding the rationals into the reals by stipulating that $p_{\mathbb{R}} = \{q \in \mathbb{Q} : q < p\}$ for each $p \in \mathbb{Q}$. We then define:

$$\alpha + \beta = \{p + q : p \in \alpha \wedge q \in \beta\}$$
$$\alpha \times \beta = \{p \times q : 0 \leq p \in \alpha \wedge 0 \leq q \in \beta\} \cup 0^{\mathbb{R}} \qquad \text{if } \alpha, \beta \geq 0_{\mathbb{R}}$$

To handle the other multiplication cases, first let:

$$-\alpha = \{p - q : p < 0 \wedge q \notin \alpha\}$$

and then stipulate:

$$\alpha \times \beta = \begin{cases} -\alpha \times -\beta & \text{if } \alpha < 0_{\mathbb{R}} \text{ and } \beta < 0_{\mathbb{R}} \\ -(-\alpha \times \beta) & \text{if } \alpha < 0_{\mathbb{R}} \text{ and } \beta > 0_{\mathbb{R}} \\ -(\alpha \times -\beta) & \text{if } \alpha > 0_{\mathbb{R}} \text{ and } \beta < 0_{\mathbb{R}} \end{cases}$$

We then need to check that each of these definitions always yields a cut. And finally, we need to go through an easy (but long-winded) demonstration that the cuts, so defined, behave exactly as they should. But we relegate all of this to section arith.6.

arith.5 Some Philosophical Reflections

sfr:arith:ref:sec

So much for the technicalities. But what did they achieve?

Well, pretty uncontestably, they gave us some lovely pure mathematics. Moreover, there were some deep conceptual achievements. It was a profound insight, to see that the Completeness Property expresses the crucial difference between the reals and the rationals. Moreover, the explicit construction of reals, as Dedekind cuts, puts the subject matter of analysis on a firm footing. We know that the notion of a *complete ordered field* is coherent, for the cuts form just such a field.

For all that, we should air a few reservations about these achievements.

First, it is not clear that thinking of reals in terms of cuts is any *more* rigorous than thinking of reals in terms of their familiar (possibly infinite) decimal expansions. This latter "construction" of the reals has some resemblance to the construction of the reals via Cauchy sequence; but in fact, it was essentially known to mathematicians from the early 17th century onwards (see section arith.7). The real increase in rigour came from the realisation that the reals have the Completeness Property; the ability to construct real numbers as particular sets is perhaps not, by itself, so very interesting.

It is even less clear that the (much easier) arithmetization of the integers, or of the rationals, increases rigour in those areas. Here, it is worth making a simple observation. Having *constructed* the integers as equivalence classes of ordered pairs of naturals, and then constructed the rationals as equivalence classes of ordered pairs of integers, and then constructed the reals as sets of rationals, we immediately *forget about* the constructions. In particular: no one would ever want to *invoke* these constructions during a mathematical proof (excepting, of course, a proof that the constructions behaved as they were supposed to). It's much easier to speak about a real, directly, than to speak about some set of sets of sets of sets of sets of sets of sets of naturals.

It is most doubtful of all that these definitions tell us what the integers, rationals, or reals *are*, *metaphysically speaking*. That is, it is doubtful that the reals (say) *are* certain sets (of sets of sets...). The main barrier to such a view is that the construction could have been done in many different ways. In the case of the reals, there are some genuinely interestingly different constructions (see section arith.7). But here is a really trivial way to obtain some different constructions: as in **??**, we could have defined ordered pairs slightly differently; if we had used this alternative notion of an ordered pair, then our constructions would have worked precisely as well as they did, but we would have ended up with different objects. As such, there are many rival set-theoretic constructions of the integers, the rationals, and the reals. And now it would just be arbitrary (and embarrassing) to claim that the integers (say) are *these* sets, rather than *those*. (As in **??**, this is an instance of an argument made famous by Benacerraf 1965.)

A further point is worth raising: there is something quite *odd* about our constructions. We started with the natural numbers. We then construct the integers, and construct "the 0 of the integers", i.e., $[0,0]_\sim$. But $0 \neq [0,0]_\sim$.

Indeed, given our constructions, *no* natural number is an integer. But that seems extremely counter-intuitive. Indeed, in **??**, we claimed without much argument that $\mathbb{N} \subseteq \mathbb{Q}$. If the constructions tell us exactly *what* the numbers are, this claim was trivially false.

Standing back, then, where do we get to? Working in a naïve set theory, and helping ourselves to the naturals, we are able to *treat* integers, rationals, and reals as certain sets. In that sense, we can *embed* the theories of these entities within a set theory. But the philosophical import of this embedding is just not that straightforward.

Of course, none of this is the last word! The point is only this. Showing that the arithmetization of the reals *is* of deep philosophical significance would require some additional *philosophical* argument.

arith.6 Ordered Rings and Fields

sfr:arith:check:sec

Throughout this chapter, we claimed that certain definitions behave "as they ought". In this technical appendix, we will spell out what we mean, and (sketch how to) show that the definitions do behave "correctly".

In section arith.1, we defined addition and multiplication on $\mathbb{Z}$. We want to show that, as defined, they endow $\mathbb{Z}$ with the structure we "would want" it to have. In particular, the structure in question is that of a commutative ring.

Definition arith.7. A *commutative ring* is a set S, equipped with specific elements 0 and 1 and operations $+$ and $\times$, satisfying these eight formulas:

Associativity	$a + (b + c) = (a + b) + c$
	$(a \times b) \times c = a \times (b \times c)$
Commutativity	$a + b = b + a$
	$a \times b = b \times a$
Identities	$a + 0 = a$
	$a \times 1 = a$
Additive Inverse	$(\exists b \in S) 0 = a + b$
Distributivity	$a \times (b + c) = (a \times b) + (a \times c)$

Implicitly, these are all bound with universal quantifiers restricted to S. And note that the elements 0 and 1 here need not be the natural numbers with the same name.

So, to check that the integers form a commutative ring, we just need to check that we meet these eight conditions. None of the conditions is difficult to establish, but this is a bit laborious. For example, here is how to prove *Associativity*, in the case of addition:

Proof. Fix $i, j, k \in \mathbb{Z}$. So there are $a_1, b_1, a_2, b_2, a_3, b_3 \in \mathbb{N}$ such that $i = [a_1, b_1]$ and $j = [a_2, b_2]$ and $k = [a_3, b_3]$. (For legibility, we write "$[x, y]$" rather than

"$[x, y]_\sim$"; we'll do this throughout this section.) Now:

$$\begin{aligned} i + (j + k) &= [a_1, b_1] + ([a_2, b_2] + [a_3, b_3]) \\ &= [a_1, b_1] + [a_2 + a_3, b_2 + b_3] \\ &= [a_1 + (a_2 + a_3), b_1 + (b_2 + b_3)] \\ &= [(a_1 + a_2) + a_3, (b_1 + b_2) + b_3] \\ &= [a_1 + a_2, b_1 + b_2] + [a_3, b_3] \\ &= ([a_1, b_1] + [a_2, b_2]) + [a_3, b_3] \\ &= (i + j) + k \end{aligned}$$

helping ourselves freely to the behavior of addition on $\mathbb{N}$. □

Equally, here is how to prove *Additive Inverse*:

Proof. Fix $i \in \mathbb{Z}$, so that $i = [a, b]$ for some $a, b \in \mathbb{N}$. Let $j = [b, a] \in \mathbb{Z}$. Helping ourselves to the behaviour of the naturals, $(a + b) + 0 = 0 + (a + b)$, so that $\langle a + b, b + a\rangle \sim_{\mathbb{Z}} \langle 0, 0\rangle$ by definition, and hence $[a + b, b + a] = [0, 0] = 0_{\mathbb{Z}}$. So now $i + j = [a, b] + [b, a] = [a + b, b + a] = [0, 0] = 0_{\mathbb{Z}}$. □

And here is a proof of *Distributivity*:

Proof. As above, fix $i = [a_1, b_1]$ and $j = [a_2, b_2]$ and $k = [a_3, b_3]$. Now:

$$\begin{aligned} i \times (j + k) &= [a_1, b_1] \times ([a_2, b_2] + [a_3, b_3]) \\ &= [a_1, b_1] \times [a_2 + a_3, b_2 + b_3] \\ &= [a_1(a_2 + a_3) + b_1(b_2 + b_3), a_1(b_2 + b_3) + b_1(a_2 + a_3)] \\ &= [a_1a_2 + a_1a_3 + b_1b_2 + b_1b_3, a_1b_2 + a_1b_3 + a_2b_1 + a_3b_1] \\ &= [a_1a_2 + b_1b_2, a_1b_2 + a_2b_1] + [a_1a_3 + b_1b_3, a_1b_3 + a_3b_1] \\ &= ([a_1, b_1] \times [a_2, b_2]) + ([a_1, b_1] \times [a_3, b_3]) \\ &= (i \times j) + (i \times k) \end{aligned}$$

□

We leave it as an exercise to prove the remaining five conditions. Having done that, we have shown that $\mathbb{Z}$ constitutes a commutative ring, i.e., that addition and multiplication (as defined) behave as they should.

Problem arith.4. Prove that $\mathbb{Z}$ is a commutative ring.

But our task is not over. As well as defining addition and multiplication over $\mathbb{Z}$, we defined an ordering relation, $\leq$, and we must check that this behaves as it should. In more detail, we must show that $\mathbb{Z}$ constitutes an *ordered* ring.[3]

[3]Recall from **??** that a total order is a relation which is reflexive, transitive, antisymmetric, and connected. In the context of order relations, connectedness is sometimes called *trichotomy*, since for any a and b we have $a \leq b \vee a = b \vee a \geq b$.

Definition arith.8. An *ordered ring* is a commutative ring which is also equipped with a total order relation, $\leq$, such that:

$$a \leq b \to a + c \leq b + c$$
$$(a \leq b \wedge 0 \leq c) \to a \times c \leq b \times c$$

Problem arith.5. Prove that $\mathbb{Z}$ is an ordered ring.

As before, it is laborious but routine to show that $\mathbb{Z}$, as constructed, is an ordered ring. We will leave that to you.

This takes care of the integers. But now we need to show very similar things of the rationals. In particular, we now need to show that the rationals form an ordered *field*, under our given definitions of $+$, $\times$, and $\leq$:

Definition arith.9. An *ordered field* is an ordered ring which also satisfies: sfr:arith:check:orderedfield

$$\textit{Multiplicative Inverse} \qquad (\forall a \in S \setminus \{0\})(\exists b \in S) a \times b = 1$$

Once you have shown that $\mathbb{Z}$ constitutes an ordered ring, it is easy but laborious to show that $\mathbb{Q}$ constitutes an ordered field.

Problem arith.6. Prove that $\mathbb{Q}$ is an ordered field.

Having dealt with the integers and the rationals, it only remains to deal with the reals. In particular, we need to show that $\mathbb{R}$ constitutes a *complete* ordered field, i.e., an ordered field with the Completeness Property. Now, Theorem arith.6 established that $\mathbb{R}$ has the Completeness Property. However, it remains to run through the (tedious) of checking that $\mathbb{R}$ is an ordered field.

Before tearing off into *that* laborious exercise, we need to check some more "immediate" things. For example, we need a guarantee that $\alpha + \beta$, as defined, is indeed a *cut*, for any cuts α and β. Here is a proof of that fact:

Proof. Since α and β are both cuts, $\alpha + \beta = \{p + q : p \in \alpha \wedge q \in \beta\}$ is a non-empty proper subset of $\mathbb{Q}$. Now suppose $x < p + q$ for some $p \in \alpha$ and $q \in \beta$. Then $x - p < q$, so $x - p \in \beta$, and $x = p + (x - p) \in \alpha + \beta$. So $\alpha + \beta$ is an initial segment of $\mathbb{Q}$. Finally, for any $p + q \in \alpha + \beta$, since α and β are both cuts, there are $p_1 \in \alpha$ and $q_1 \in \beta$ such that $p < p_1$ and $q < q_1$; so $p + q < p_1 + q_1 \in \alpha + \beta$; so $\alpha + \beta$ has no maximum. □

Similar efforts will allow you to check that $\alpha - \beta$ and $\alpha \times \beta$ and $\alpha \div \beta$ are cuts (in the last case, ignoring the case where β is the zero-cut). Again, though, we will simply leave this to you.

Problem arith.7. Prove that $\mathbb{R}$ is an ordered field.

But here is a small loose end to tidy up. In section arith.4, we suggest that we can take $\sqrt{2} = \{p \in \mathbb{Q} : p < 0 \text{ or } p^2 < 2\}$. But we do need to show that this set is a *cut*. Here is a proof of that fact:

Proof. Clearly this is a nonempty proper initial segment of the rationals; so it suffices to show that it has no maximum. In particular, it suffices to show that, where p is a positive rational with $p^2 < 2$ and $q = \frac{2p+2}{p+2}$, both $p < q$ and $q^2 < 2$. To see that $p < q$, just note:

$$\begin{aligned} p^2 &< 2 \\ p^2 + 2p &< 2 + 2p \\ p(p+2) &< 2 + 2p \\ p &< \tfrac{2+2p}{p+2} = q \end{aligned}$$

To see that $q^2 < 2$, just note:

$$\begin{aligned} p^2 &< 2 \\ 2p^2 + 4p + 2 &< p^2 + 4p + 4 \\ 4p^2 + 8p + 4 &< 2(p^2 + 4p + 4) \\ (2p+2)^2 &< 2(p+2)^2 \\ \tfrac{(2p+2)^2}{(p+2)^2} &< 2 \\ q^2 &< 2 \end{aligned}$$

□

arith.7 Appendix: the Reals as Cauchy Sequences

sfr:arith:cauchy:sec

In section arith.4, we constructed the reals as Dedekind cuts. In this section, we explain an alternative construction. It builds on Cauchy's definition of (what we now call) a Cauchy sequence; but the use of this definition to *construct* the reals is due to other nineteenth-century authors, notably Weierstrass, Heine, Méray and Cantor. (For a nice history, see O'Connor and Robertson 2005.)

Before we get to the nineteenth century, it's worth considering Simon Stevin (1548–1620). In brief, Stevin realised that we can think of each real in terms of its decimal expansion. Thus even an irrational number, like $\sqrt{2}$, has a nice decimal expansion, beginning:

$$1.41421356237\ldots$$

It is very easy to model decimal expansions in set theory: simply consider them as functions $d\colon \mathbb{N} \to \mathbb{N}$, where $d(n)$ is the nth decimal place that we are interested in. We will then need a bit of tweak, to handle the bit of the real number that comes before the decimal point (here, just 1). We will also need a further tweak (an equivalence relation) to guarantee that, for example, $0.999\ldots = 1$. But it is not difficult to offer a perfectly rigorous construction of the real numbers, in the manner of Stevin, within set theory.

Stevin is not our focus. (For more on Stevin, see Katz and Katz 2012.) But here is a closely related thought. Instead of treating $\sqrt{2}$'s decimal expansion

directly, we can instead consider a *sequence* of increasingly accurate rational approximations to $\sqrt{2}$, by considering the increasingly precise expansions:

$$1, 1.4, 1.414, 1.4142, 1.41421, \ldots$$

The idea that reals can be considered via "increasingly good approximations" provides us with the basis for another sequence of insights (akin to the realisations that we used when constructing $\mathbb{Q}$ from $\mathbb{Z}$, or $\mathbb{Z}$ from $\mathbb{N}$). The basic insights are these:

1. Every real can be written as a (perhaps infinite) decimal expansion.
2. The information encoded by a (perhaps infinite) decimal expansion can be equally be encoded by a sequence of rational numbers.
3. A sequence of rational numbers can be thought of as a function from $\mathbb{N}$ to $\mathbb{Q}$; just let $f(n)$ be the nth rational in the sequence.

Of course, not just *any* function from $\mathbb{N}$ to $\mathbb{Q}$ will give us a real number. For instance, consider this function:

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Essentially the worry here is that the sequence $0, 1, 0, 1, 0, 1, 0, \ldots$ doesn't seem to "hone in" on any real. So: to ensure that we consider sequences which do hone in on some real, we need to restrict our attention to sequences which have some *limit*.

We have already encountered the idea of a limit, in **??**. But we cannot use *quite* the same definition as we used there. The expression "$(\forall \varepsilon > 0)$" there tacitly involved quantification over the real numbers; and we were considering the limits of functions on the real numbers; so invoking that definition would be to help ourselves to the real numbers; and they are exactly what we were aiming to *construct*. Fortunately, we can work with a closely related idea of a limit.

Definition arith.10. A function $f : \mathbb{N} \to \mathbb{Q}$ is a *Cauchy sequence* iff for any positive $\varepsilon \in \mathbb{Q}$ we have that $(\exists \ell \in \mathbb{N})(\forall m, n > \ell)|f(m) - f(n)| < \varepsilon$. sfr:arith:cauchy:def:CauchySequence

The general idea of a limit is the same as before: if you want a certain level of precision (measured by ε), there is a "region" to look in (any input greater than ℓ). And it is easy to see that our sequence 1, 1.4, 1.414, 1.4142, 1.41421... has a limit: if you want to approximate $\sqrt{2}$ to within an error of $1/10^n$, then just look to any entry after the nth.

The obvious thought, then, would be to say that a real number just *is* any Cauchy sequence. But, as in the constructions of $\mathbb{Z}$ and $\mathbb{Q}$, this would be too naïve: for any given real number, multiple different Cauchy sequences indicate that real number. A simple way to see this as follows. Given a Cauchy

sequence f, define g to be exactly the same function as f, except that $g(0) \neq f(0)$. Since the two sequences agree everywhere after the first number, we will (ultimately) want to say that they have the same limit, in the sense employed in Definition arith.10, and so should be thought of "defining" the same real. So, we should really think of these Cauchy sequences as the same real number.

Consequently, we again need to define an equivalence relation on the Cauchy sequences, and identify real numbers with equivalence relations. First we need the idea of a function which tends to 0 in the limit. For any function $h : \mathbb{N} \to \mathbb{Q}$, say that h *tends to* 0 iff for any positive $\varepsilon \in \mathbb{Q}$ we have that $(\exists \ell \in \mathbb{N})(\forall n > \ell)|f(n)| < \varepsilon$.[4] Further, where f and g are functions $\mathbb{N} \to \mathbb{Q}$, let $(f - g)(n) = f(n) - g(n)$. Now define:

$$f \eqsim g \text{ iff } (f - g) \text{ tends to } 0.$$

We need to check that $\eqsim$ is an equivalence relation; and it is. We can then, if we like, define the reals as the equivalence classes, under $\eqsim$, of all Cauchy sequences from $\mathbb{N} \to \mathbb{Q}$.

Problem arith.8. Let $f(n) = 0$ for every n. Let $g(n) = \frac{1}{(n+1)^2}$. Show that both are Cauchy sequences, and indeed that the limit of both functions is 0, so that also $f \sim_{\mathbb{R}} g$.

Having done this, we shall as usual write $[f]_{\eqsim}$ for the equivalence class with f as an element. However, to keep things readable, in what follows we will drop the subscript and write just $[f]$. We also stipulate that, for each $q \in \mathbb{Q}$, we have $q_{\mathbb{R}} = [c_q]$, where c_q is the constant function $c_q(n) = q$ for all $n \in \mathbb{N}$. We then define basic relations and operations on the reals, e.g.:

$$[f] + [g] = [(f + g)]$$
$$[f] \times [g] = [(f \times g)]$$

where $(f+g)(n) = f(n)+g(n)$ and $(f \times g)(n) = f(n) \times g(n)$. Of course, we also need to check that each of $(f + g)$, $(f - g)$ and $(f \times g)$ are Cauchy sequences when f and g are; but they are, and we leave this to you.

Finally, we define we a notion of order. Say $[f]$ is *positive* iff both $[f] \neq 0_{\mathbb{Q}}$ and $(\exists \ell \in \mathbb{N})(\forall n > \ell)0 < f(n)$. Then say $[f] < [g]$ iff $[(g - f)]$ is positive. We have to check that this is well-defined (i.e., that it does not depend upon choice of "representative" function from the equivalence class). But having done this, it is quite easy to show that these yield the right algebraic properties; that is:

sfr:arith:cauchy: thm:cauchyorderedfield

Theorem arith.11. *The Cauchy sequences constitute an ordered field.*

Proof. Exercise. □

Problem arith.9. Prove that the Cauchy sequences constitute an ordered field.

[4] Compare this with the definition of $\lim_{x\to\infty} f(x) = 0$ in **??**.

It is harder to prove that the reals, so constructed, have the Completeness Property, so we will give the proof.

Theorem arith.12. *Every non-empty set of Cauchy sequences with an upper bound has a least upper bound.*

Proof sketch. Let S be any non-empty set of Cauchy sequences with an upper bound. So there is some $p \in \mathbb{Q}$ such that $p_{\mathbb{R}}$ is an upper bound for S. Let $r \in S$; then there is some $q \in \mathbb{Q}$ such that $q_{\mathbb{R}} < r$. So if a least upper bound on S exists, it is between $q_{\mathbb{R}}$ and $p_{\mathbb{R}}$ (inclusive).

We will hone in on the l.u.b., by approaching it simultaneously from below and above. In particular, we define two functions, $f, g\colon \mathbb{N} \to \mathbb{Q}$, with the aim that f will hone in on the l.u.b. from above, and g will hone on in it from below. We start by defining:

$$f(0) = p$$
$$g(0) = q$$

Then, where $a_n = \frac{f(n)+g(n)}{2}$, let:[5]

$$f(n+1) = \begin{cases} a_n & \text{if } (\forall h \in S)[h] \leq (a_n)_{\mathbb{R}} \\ f(n) & \text{otherwise} \end{cases}$$
$$g(n+1) = \begin{cases} a_n & \text{if } (\exists h \in S)[h] \geq (a_n)_{\mathbb{R}} \\ g(n) & \text{otherwise} \end{cases}$$

Both f and g are Cauchy sequences. (This can be checked fairly easily; but we leave it as an exercise.) Note that the function $(f - g)$ tends to 0, since the difference between f and g halves at every step. Hence $[f] = [g]$.

We will show that $(\forall h \in S)[h] \leq [f]$, invoking Theorem arith.11 as we go. Let $h \in S$ and suppose, for reductio, that $[f] < [h]$, so that $0_{\mathbb{R}} < [(h - f)]$. Since f is a monotonically decreasing Cauchy sequence, there is some $n \in \mathbb{N}$ such that $[(c_{f(n)} - f)] < [(h - f)]$. So:

$$(f(n))_{\mathbb{R}} = [c_{f(k)}] < [f] + [(h - f)] = [h],$$

contradicting the fact that, by construction, $[h] \leq (f(k))_{\mathbb{R}}$.

In an exactly similar way, we can show that $(\forall [h]_{\in} S)[g] \leq [h]$. So $[f] = [g]$ is the *least* upper bound for S. □

[5] This is a recursive definition. But we have not *yet* given any reason to think that recursive definitions are ok.

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