

Chapter udf

Steps towards Z

z.1 The Story in More Detail

sth:z:story:
sec In ??, we quoted Schoenfield’s description of the process of set-formation. We now want to write down a few more principles, to make this story a bit more precise. Here they are:

Stages-are-key. Every set is formed at some stage.

Stages-are-ordered. Stages are ordered: some come *before* others.¹

Stages-accumulate. For any stage S , and for any sets which were formed *before* stage S : a set is formed at stage S whose members are exactly those sets. Nothing else is formed at stage S .

These are informal principles, but we will be able to use them to vindicate several of the axioms of Zermelo’s set theory.

(We should offer a word of caution. Although we will be presenting some completely standard axioms, with completely standard names, the italicized principles we have just presented have no particular names in the literature. We simply monikers which we hope are helpful.)

z.2 Separation

sth:z:sep:
sec We start with a principle to replace Naïve Comprehension:

Axiom (Scheme of Separation). For every formula $\varphi(x)$, this is an axiom: for any A , the set $\{x \in A : \varphi(x)\}$ exists.

¹We will actually assume—tacitly—that the stages are *well-ordered*. What this amounts to is explained in ??. This is a substantial assumption. In fact, using a very clever technique due to [Scott \(1974\)](#), this assumption can be *avoided* and then *derived*. (This will also explain why we should think that there is an initial stage.) We cannot go into that here; for more, see [Button \(2021\)](#).

Note that this is not a single axiom. It is a *scheme* of axioms. There are *infinitely many* Separation axioms; one for every formula $\varphi(x)$. The scheme can equally well be (and normally is) written down as follows:

For any formula $\varphi(x)$ which does not contain “ S ”, this is an axiom:

$$\forall A \exists S \forall x (x \in S \leftrightarrow (\varphi(x) \wedge x \in A)).$$

In keeping with the convention noted at the start of ??, the formulas φ in the Separation axioms may have parameters.²

Separation is immediately justified by our cumulative-iterative conception of sets we have been telling. To see why, let A be a set. So A is formed by some stage S (by *Stages-are-key*). Since A was formed at stage S , all of A ’s members were formed before stage S (by *Stages-accumulate*). Now in particular, consider all the sets which are members of A and which also satisfy φ ; clearly all of these sets, too, were formed before stage S . So they are formed into a set $\{x \in A : \varphi(x)\}$ at stage S too (by *Stages-accumulate*).

Unlike Naïve Comprehension, this avoid Russell’s Paradox. For we cannot simply assert the existence of the set $\{x : x \notin x\}$. Rather, *given* some set A , we can assert the existence of the set $R_A = \{x \in A : x \notin x\}$. But all this proves is that $R_A \notin R_A$ and $R_A \notin A$, none of which is very worrying.

However, Separation has an immediate and striking consequence:

Theorem z.1. *There is no universal set, i.e., $\{x : x = x\}$ does not exist.*

*sth:z:sep:
thm:NoUniversalSet*

Proof. For reductio, suppose V is a universal set. Then by Separation, $R = \{x \in V : x \notin x\} = \{x : x \notin x\}$ exists, contradicting Russell’s Paradox. \square

The absence of a universal set—indeed, the open-endedness of the hierarchy of sets—is one of the most fundamental ideas behind the cumulative-iterative conception. So it is worth seeing that, intuitively, we could reach it via a different route. A universal set must be **an element** of itself. But, on our cumulative-iterative conception, every set appears (for the first time) in the hierarchy at the first stage immediately after all of its **elements**. But this entails that *no* set is self-membered. For any self-membered set would have to first occur immediately after the stage at which it first occurred, which is absurd. (We will see in ?? how to make this explanation more rigorous, by using the notion of the “rank” of a set. However, we will need to have a few more axioms in place to do this.)

Here are a few more consequences of Separation and Extensionality.

Proposition z.2. *If any set exists, then \emptyset exists.*

*sth:z:sep:
prop:emptyexists*

Proof. If A is a set, $\emptyset = \{x \in A : x \neq x\}$ exists by Separation. \square

²For an explanation of what this means, see the discussion immediately after ??.

Proposition z.3. $A \setminus B$ exists for any sets A and B

Proof. $A \setminus B = \{x \in A : x \notin B\}$ exists by Separation. \square

It also turns out that (almost) arbitrary intersections exist:

sth:z:sep:
prop:intersectionsexist

Proposition z.4. If $A \neq \emptyset$, then $\bigcap A = \{x : (\forall y \in A)x \in y\}$ exists.

Proof. Let $A \neq \emptyset$, so there is some $c \in A$. Then $\bigcap A = \{x : (\forall y \in A)x \in y\} = \{x \in c : (\forall y \in A)x \in y\}$, which exists by Separation. \square

Note the condition that $A \neq \emptyset$, though; for $\bigcap \emptyset$ would be the universal set, vacuously, contradicting **Theorem z.1**.

z.3 Union

sth:z:union:
sec

Proposition z.4 gave us intersections. But if we want arbitrary unions to exist, we need to lay down another axiom:

Axiom (Union). For any set A , the set $\bigcup A = \{x : (\exists b \in A)x \in b\}$ exists.

$$\forall A \exists U \forall x (x \in U \leftrightarrow (\exists b \in A)x \in b)$$

This axiom is also justified by the cumulative-iterative conception. Let A be a set, so A is formed at some stage S (by *Stages-are-key*). Every member of A was formed *before* S (by *Stages-accumulate*); so, reasoning similarly, every member of every member of A was formed before S . Thus all of *those* sets are available before S , to be formed into a set at S . And that set is just $\bigcup A$.

z.4 Pairs

sth:z:pairs:
sec

The next axiom to consider is the following:

Axiom (Pairs). For any sets a, b , the set $\{a, b\}$ exists.

$$\forall a \forall b \exists P \forall x (x \in P \leftrightarrow (x = a \vee x = b))$$

Here is how to justify this axiom, using the iterative conception. Suppose a is available at stage S , and b is available at stage T . Let M be whichever of stages S and T comes later. Then since a and b are both available at stage M , the set $\{a, b\}$ is a possible collection available at any stage after M (whichever is the greater).

But hold on! Why assume that there *are* any stages after M ? If there are none, then our justification will fail. So, to justify Pairs, we will have to add another principle to the story we told in **section z.1**, namely:

Stages-keep-going. There is no last stage.

Is this principle justified? Nothing in Shoenfield's story stated *explicitly* that there is no last stage. Still, even if it is (strictly speaking) an extra addition to our story, it fits well with the basic idea that sets are formed in stages. We will simply accept it in what follows. And so, we will accept the Axiom of Pairs too.

Armed with this new Axiom, we can prove the existence of plenty more sets. For example:

Proposition z.5. *For any sets a and b , the following sets exist:*

1. $\{a\}$
2. $a \cup b$
3. $\langle a, b \rangle$

*sth:z:pairs:
prop:pairsconsequences
sth:z:pairs:
singleton
sth:z:pairs:
binunion
sth:z:pairs:
tuples*

Proof. (1). By Pairs, $\{a, a\}$ exists, which is $\{a\}$ by Extensionality.

(2). By Pairs, $\{a, b\}$ exists. Now $a \cup b = \bigcup\{a, b\}$ exists by Union.

(3). By (1), $\{a\}$ exists. By Pairs, $\{a, b\}$ exists. Now $\{\{a\}, \{a, b\}\} = \langle a, b \rangle$ exists, by Pairs again. \square

Problem z.1. Show that, for any sets a, b, c , the set $\{a, b, c\}$ exists.

Problem z.2. Show that, for any sets a_1, \dots, a_n , the set $\{a_1, \dots, a_n\}$ exists.

z.5 Powersets

We will proceed with another axiom:

*sth:z:power:
sec*

Axiom (Powersets). For any set A , the set $\wp(A) = \{x : x \subseteq A\}$ exists.

$$\forall A \exists P \forall x (x \in P \leftrightarrow (\forall z \in x) z \in A)$$

Our justification for this is pretty straightforward. Suppose A is formed at stage S . Then all of A 's members were available before S (by *Stages-accumulate*). So, reasoning as in our justification for Separation, every subset of A is formed by stage S . So they are all available, to be formed into a single set, at any stage after S . And we know that there is some such stage, since S is not the last stage (by *Stages-keep-going*). So $\wp(A)$ exists.

Here is a nice consequence of Powersets:

Proposition z.6. *Given any sets A, B , their Cartesian product $A \times B$ exists.*

Proof. The set $\wp(\wp(A \cup B))$ exists by Powersets and **Proposition z.5**. So by Separation, this set exists:

$$C = \{z \in \wp(\wp(A \cup B)) : (\exists x \in A)(\exists y \in B)z = \langle x, y \rangle\}.$$

Now, for any $x \in A$ and $y \in B$, the set $\langle x, y \rangle$ exists by **Proposition z.5**. Moreover, since $x, y \in A \cup B$, we have that $\{x\}, \{x, y\} \in \wp(A \cup B)$, and $\langle x, y \rangle \in \wp(\wp(A \cup B))$. So $A \times B = C$. \square

In this proof, Powerset interacts with Separation. And that is no surprise. Without Separation, Powersets wouldn't be a very *powerful* principle. After all, Separation tells us which subsets of a set exist, and hence determines just how “fat” each Powerset is.

Problem z.3. Show that, for any sets A, B : (i) the set of all relations with domain A and range B exists; and (ii) the set of all functions from A to B exists.

Problem z.4. Let A be a set, and let \sim be an equivalence relation on A . Prove that the set of equivalence classes under \sim on A , i.e., A/\sim , exists.

z.6 Infinity

sth:z:infinity-again:
sec

We already have enough axioms to ensure that there are infinitely many sets (if there are any). For suppose some set exists, and so \emptyset exists (by [Proposition z.2](#)). Now for any set x , the set $x \cup \{x\}$ exists by [Proposition z.5](#). So, applying this a few times, we will get sets as follows:

0. \emptyset
1. $\{\emptyset\}$
2. $\{\emptyset, \{\emptyset\}\}$
3. $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$
4. $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$

and we can check that each of these sets is distinct.

We have started the numbering from 0, for a few reasons. But one of them is this. It is not that hard to check that the set we have labelled “ n ” has exactly n members, and (intuitively) is formed at the n th stage.

But. This gives us *infinitely many* sets, but it does not guarantee that there is an *infinite set*, i.e., a set with infinitely many members. And this really matters: unless we can find a (Dedekind) infinite set, we cannot construct a Dedekind algebra. But we want a Dedekind algebra, so that we can treat it as the set of natural numbers. (Compare ??.)

Importantly, the axioms we have laid down so far do *not* guarantee the existence of any infinite set. So we have to lay down a new axiom:

Axiom (Infinity). There is a set, I , such that $\emptyset \in I$ and $x \cup \{x\} \in I$ whenever $x \in I$.

$$\exists I((\emptyset \in I) \wedge \forall x (x \in I \rightarrow x \cup \{x\} \in I) \wedge \\ (\forall x \in I)(\exists s \in I) \forall z (z \in s \leftrightarrow (z \in x \vee z = x)))$$

It is easy to see that the set I given to us by the Axiom of Infinity is Dedekind infinite. Its distinguished element is \emptyset , and the injection on I is given by $s(x) = x \cup \{x\}$. Now, ?? showed how to extract a Dedekind Algebra from a Dedekind infinite set; and we will treat this as our set of natural numbers. More precisely:

Definition z.7. Let I be any set given to us by the Axiom of Infinity. Let s be the function $s(x) = x \cup \{x\}$. Let $\omega = \text{clo}_s(\emptyset)$. We call the members of ω the *natural numbers*, and say that n is the result of n -many applications of s to \emptyset .

sth:z:infinity-again:
defnomega

You can now look back and check that the set labelled “ n ”, a few paragraphs earlier, will be treated *as* the number n .

We will discuss this significance of this stipulation in [section z.8](#). For now, it enables us to prove an intuitive result:

Proposition z.8. *No natural number is Dedekind infinite.*

sth:z:infinity-again:
naturalnumbersarentinfinite

Proof. The proof is by induction, i.e., ??. Clearly $0 = \emptyset$ is not Dedekind infinite. For the induction step, we will establish the contrapositive: if (absurdly) $s(n)$ is Dedekind infinite, then n is Dedekind infinite.

So suppose that $s(n)$ is Dedekind infinite, i.e., there is some *injection* f with $\text{ran}(f) \subsetneq \text{dom}(f) = s(n) = n \cup \{n\}$. There are two cases to consider.

Case 1: $n \notin \text{ran}(f)$. So $\text{ran}(f) \subseteq n$, and $f(n) \in n$. Let $g = f \upharpoonright_n$; now $\text{ran}(g) = \text{ran}(f) \setminus \{f(n)\} \subsetneq n = \text{dom}(g)$. Hence n is Dedekind infinite.

Case 2: $n \in \text{ran}(f)$. Fix $m \in \text{dom}(f) \setminus \text{ran}(f)$, and define a function h with domain $s(n) = n \cup \{n\}$:

$$h(x) = \begin{cases} f(x) & \text{if } f(x) \neq n \\ m & \text{if } f(x) = n \end{cases}$$

So h and f agree everywhere, except that $h(f^{-1}(n)) = m \neq n = f(f^{-1}(n))$. Since f is an *injection*, $n \notin \text{ran}(h)$; and $\text{ran}(h) \subsetneq \text{dom}(h) = s(n)$. Now n is Dedekind infinite, using the argument of Case 1. \square

The question remains, though, of how we might *justify* the Axiom of Infinity. The short answer is that we will need to add another principle to the story we have been telling. That principle is as follows:

Stages-hit-infinity. There is an infinite stage. That is, there is a stage which (a) is not the first stage, and which (b) has some stages before it, but which (c) has no immediate predecessor.

The Axiom of Infinity follows straightforwardly from this principle. We know that natural number n is formed at stage n . So the set ω is formed at the first infinite stage. And ω itself witnesses the Axiom of Infinity.

This, however, simply pushes us back to the question of how we might justify *Stages-hit-infinity*. As with *Stages-keep-going*, it was not an explicit part of the story we told about the cumulative-iterative hierarchy. But more

than that: nothing in the very idea of an iterative hierarchy, in which sets are formed stage by stage, forces us to think that the process involves an *infinite* stage. It seems perfectly coherent to think that the stages are ordered like the natural numbers.

This, however, gives rise to an obvious problem. In ??, we considered Dedekind’s “proof” that there is a Dedekind infinite set (of thoughts). This may not have struck you as very satisfying. But if *Stages-hit-infinity* is not “forced upon us” by the iterative conception of set (or by “the laws of thought”), then we are still left without an intrinsic justification for the claim that there is a Dedekind infinite set.

There is much more to say here, of course. But hopefully you are now at a point to start thinking about what it might *take* to justify an axiom (or principle). In what follows we will simply take *Stages-hit-infinity* for granted.

z.7 \mathbf{Z}^- : a Milestone

sth:z:milestone:
sec We will revisit *Stages-hit-infinity* in the next section. However, with the Axiom of Infinity, we have reached an important milestone. We now have all the axioms required for the theory \mathbf{Z}^- . In detail:

Definition z.9. The theory \mathbf{Z}^- has these axioms: Extensionality, Union, Pairs, Powersets, Infinity, and all instances of the Separation scheme.

The name stands for *Zermelo* set theory (*minus* something which we will come to later). Zermelo deserves the honour, since he essentially formulated this theory in his 1908.³

This theory is powerful enough to allow us to do an enormous amount of mathematics. In particular, you *should* look back through ??, and convince yourself that everything we did, naïvely, could be done more formally within \mathbf{Z}^- . (Once you have done that for a bit, you might want to skip ahead and read [section z.9](#).) So, henceforth, and without any further comment, we will take ourselves to be working in \mathbf{Z}^- (at least).

z.8 Selecting our Natural Numbers

sth:z:nat:
sec In [Definition z.7](#), we explicitly defined the expression “natural numbers”. How should you understand this stipulation? It is not a metaphysical claim, but just a decision to *treat* certain sets as the natural numbers. We touched upon reasons for thinking this in ??, ?? and ??. But we can make these reasons even more pointed.

Our Axiom of Infinity follows [von Neumann \(1925\)](#). But here is another axiom, which we could have adopted instead:

³For interesting comments on the history and technicalities, see [Potter \(2004, Appendix A\)](#).

Zermelo’s 1908 Axiom of Infinity. There is a set A such that $\emptyset \in A$ and $(\forall x \in A)\{x\} \in A$.

Had we used Zermelo’s axiom, instead of our (von Neumann-inspired) Axiom of Infinity, we would equally well have been given a Dedekind infinite set, and so a Dedekind algebra. On Zermelo’s approach, the distinguished element of our algebra would again have been \emptyset (our surrogate for 0), but the injection would have been given by the map $x \mapsto \{x\}$, rather than $x \mapsto x \cup \{x\}$. The simplest upshot of this is that Zermelo treats 2 as $\{\{\emptyset\}\}$, whereas we (with von Neumann) treat 2 as $\{\emptyset, \{\emptyset\}\}$.

Why choose one axiom of Infinity rather than the other? The main practical reason is that von Neumann’s approach “scales up” to handle transfinite numbers rather well. We will explore this from ?? onwards. However, from the simple perspective of *doing arithmetic*, both approaches would do equally well. So if someone tells you that the natural numbers *are* sets, the obvious question is: *Which sets are they?*

This precise question was made famous by Benacerraf (1965). But it is worth emphasising that it is just the most famous example of a phenomenon that we have encountered many times already. The basic point is this. Set theory gives us a way to *simulate* a bunch of “intuitive” kinds of entities: the reals, rationals, integers, and naturals, yes; but also ordered pairs, functions, and relations. However, set theory never provides us with a *unique* choice of simulation. There are *always* alternatives which—straightforwardly—would have served us just as well.

z.9 Appendix: Closure, Comprehension, and Intersection

In section z.7, we suggested that you should look back through the naïve work of ?? and check that it can be carried out in \mathbf{Z}^- . If you followed that advice, *one* point might have tripped you up: the use of *intersection* in Dedekind’s treatment of *closures*.

[sth:z:arbintersections:sec](#)

Recall from ?? that

$$\text{clo}_f(o) = \bigcap \{X : o \in X \text{ and } X \text{ is } f\text{-closed}\}.$$

The general shape of this is a definition of the form:

$$C = \bigcap \{X : \varphi(X)\}.$$

But this should ring alarm bells: since Naïve Comprehension fails, there is no guarantee that $\{X : \varphi(X)\}$ exists. It looks dangerously, then, like such definitions are *cheating*.

Fortunately, they are not cheating; or rather, if they *are* cheating as they stand, then we can engage in some honest toil to render them kosher. That honest toil was foreshadowed in [Proposition z.4](#), when we explained why $\bigcap A$ exists for any $A \neq \emptyset$. But we will spell it out explicitly.

Given Extensionality, if we attempt to define C as $\bigcap\{X : \varphi(X)\}$, all we are really asking is for an object C which obeys the following:

$$\forall x(x \in C \leftrightarrow \forall X(\varphi(X) \rightarrow x \in X)) \tag{*}$$

[sth:zarbintersections:](#)
[bicondelimarbintersection](#)

Now, suppose there is *some* set, S , such that $\varphi(S)$. Then to deliver [eq. \(*\)](#), we can simply define C using *Separation*, as follows:

$$C = \{x \in S : \forall X(\varphi(X) \rightarrow x \in X)\}.$$

We leave it as an exercise to check that this definition yields [eq. \(*\)](#), as desired. And this general strategy will allow us to circumvent any apparent use of Naïve Comprehension in defining intersections. In the particular case which got us started on this line of thought, namely that of $\text{clo}_f(o)$, here is how that would work. We began the proof of [??](#) by noting that $o \in \text{ran}(f) \cup \{o\}$ and that $\text{ran}(f) \cup \{o\}$ is f -closed. So, we can define what we want thus:

$$\text{clo}_f(o) = \{x \in \text{ran}(f) \cup \{o\} : (\forall X \ni o)(X \text{ is } f\text{-closed} \rightarrow x \in X)\}.$$

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