

z.1 Separation

sth:z:sep:
sec We start with a principle to replace Naïve Comprehension:

Axiom (Scheme of Separation). For every formula $\varphi(x)$, this is an axiom: for any A , the set $\{x \in A : \varphi(x)\}$ exists.

Note that this is not a single axiom. It is a *scheme* of axioms. There are *infinitely many* Separation axioms; one for every formula $\varphi(x)$. The scheme can equally well be (and normally is) written down as follows:

For any formula $\varphi(x)$ which does not contain “ S ”, this is an axiom:

$$\forall A \exists S \forall x (x \in S \leftrightarrow (\varphi(x) \wedge x \in A)).$$

In keeping with the convention noted at the start of ??, the formulas φ in the Separation axioms may have parameters.¹

Separation is immediately justified by our cumulative-iterative conception of sets we have been telling. To see why, let A be a set. So A is formed by some stage S (by *Stages-are-key*). Since A was formed at stage S , all of A ’s members were formed before stage S (by *Stages-accumulate*). Now in particular, consider all the sets which are members of A and which also satisfy φ ; clearly all of these sets, too, were formed before stage S . So they are formed into a set $\{x \in A : \varphi(x)\}$ at stage S too (by *Stages-accumulate*).

Unlike Naïve Comprehension, this avoid Russell’s Paradox. For we cannot simply assert the existence of the set $\{x : x \notin x\}$. Rather, *given* some set A , we can assert the existence of the set $R_A = \{x \in A : x \notin x\}$. But all this proves is that $R_A \notin R_A$ and $R_A \notin A$, none of which is very worrying.

However, Separation has an immediate and striking consequence:

sth:z:sep:
thm:NoUniversalSet **Theorem z.1.** *There is no universal set, i.e., $\{x : x = x\}$ does not exist.*

Proof. For reductio, suppose V is a universal set. Then by Separation, $R = \{x \in V : x \notin x\} = \{x : x \notin x\}$ exists, contradicting Russell’s Paradox. \square

The absence of a universal set—indeed, the open-endedness of the hierarchy of sets—is one of the most fundamental ideas behind the cumulative-iterative conception. So it is worth seeing that, intuitively, we could reach it via a different route. A universal set must be an **element** of itself. But, on our cumulative-iterative conception, every set appears (for the first time) in the hierarchy at the first stage immediately after all of its **elements**. But this entails that *no* set is self-membered. For any self-membered set would have to first occur immediately after the stage at which it first occurred, which is

¹For an explanation of what this means, see the discussion immediately after ??.

absurd. (We will see in ?? how to make this explanation more rigorous, by using the notion of the “rank” of a set. However, we will need to have a few more axioms in place to do this.)

Here are a few more consequences of Separation and Extensionality.

Proposition z.2. *If any set exists, then \emptyset exists.*

*sth:z:sep:
prop:emptyexists*

Proof. If A is a set, $\emptyset = \{x \in A : x \neq x\}$ exists by Separation. □

Proposition z.3. *$A \setminus B$ exists for any sets A and B*

Proof. $A \setminus B = \{x \in A : x \notin B\}$ exists by Separation. □

It also turns out that (almost) arbitrary intersections exist:

Proposition z.4. *If $A \neq \emptyset$, then $\bigcap A = \{x : (\forall y \in A)x \in y\}$ exists.*

*sth:z:sep:
prop:intersectionexist*

Proof. Let $A \neq \emptyset$, so there is some $c \in A$. Then $\bigcap A = \{x : (\forall y \in A)x \in y\} = \{x \in c : (\forall y \in A)x \in y\}$, which exists by Separation. □

Note the condition that $A \neq \emptyset$, though; for $\bigcap \emptyset$ would be the universal set, vacuously, contradicting **Theorem z.1**.

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Bibliography