z.1 Powersets

We will proceed with another axiom:

**Axiom (Powersets).** For any set \( A \), the set \( \wp(A) = \{ x : x \subseteq A \} \) exists.

\[
\forall A \exists P \forall x (x \in P \leftrightarrow (\forall z \in x) z \in A)
\]

Our justification for this is pretty straightforward. Suppose \( A \) is formed at stage \( S \). Then all of \( A \)'s members were available before \( S \) (by Stages-accumulate). So, reasoning as in our justification for Separation, every subset of \( A \) is formed by stage \( S \). So they are all available, to be formed into a single set, at any stage after \( S \). And we know that there is some such stage, since \( S \) is not the last stage (by Stages-keep-going). So \( \wp(A) \) exists.

Here is a nice consequence of Powersets:

**Proposition z.1.** Given any sets \( A, B \), their Cartesian product \( A \times B \) exists.

**Proof.** The set \( \wp(\wp(A \cup B)) \) exists by Powersets and ???. So by Separation, this set exists:

\[
C = \{ z \in \wp(\wp(A \cup B)) : (\exists x \in A)(\exists y \in B) z = \langle x, y \rangle \}.
\]

Now, for any \( x \in A \) and \( y \in B \), the set \( \langle x, y \rangle \) exists by ???. Moreover, since \( x, y \in A \cup B \), we have that \( \{ x \}, \{ x, y \} \in \wp(A \cup B) \), and \( \langle x, y \rangle \in \wp(\wp(A \cup B)) \). So \( A \times B = C \).

In this proof, Powerset interacts with Separation. And that is no surprise. Without Separation, Powersets wouldn’t be a very powerful principle. After all, Separation tells us which subsets of a set exist, and hence determines just how “fat” each Powerset is.

**Problem z.1.** Show that, for any sets \( A, B \): (i) the set of all relations with domain \( A \) and range \( B \) exists; and (ii) the set of all functions from \( A \) to \( B \) exists.

**Problem z.2.** Let \( A \) be a set, and let \( \sim \) be an equivalence relation on \( A \). Prove that the set of equivalence classes under \( \sim \) on \( A \), i.e., \( A/\sim \), exists.

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Bibliography