z.1 Powersets

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We will proceed with another axiom:

Axiom (Powersets)**.** For any set A, the set $\wp(A) = \{x : x \subseteq A\}$ exists.
$\forall A \exists P \forall x(x \in P \leftrightarrow (\forall z \in x) z \in A)$

Our justification for this is pretty straightforward. Suppose A is formed at stage S. Then all of A's members were available before S (by *Stages-accumulate*). So, reasoning as in our justification for Separation, every subset of A is formed by stage S. So they are all available, to be formed into a single set, at any stage after S. And we know that there is some such stage, since S is not the last stage (by *Stages-keep-going*). So $\wp(A)$ exists.

Here is a nice consequence of Powersets:

Proposition z.1. *Given any sets A, B, their Cartesian product $A \times B$ exists.*

Proof. The set $\wp(\wp(A \cup B))$ exists by Powersets and **??**. So by Separation, this set exists:

$$C = \{z \in \wp(\wp(A \cup B)) : (\exists x \in A)(\exists y \in B) z = \langle x, y \rangle\}.$$

Now, for any $x \in A$ and $y \in B$, the set $\langle x, y \rangle$ exists by **??**. Moreover, since $x, y \in A \cup B$, we have that $\{x\}, \{x, y\} \in \wp(A \cup B)$, and $\langle x, y \rangle \in \wp(\wp(A \cup B))$. So $A \times B = C$. □

In this proof, Powerset interacts with Separation. And that is no surprise. Without Separation, Powersets wouldn't be a very *powerful* principle. After all, Separation tells us which subsets of a set exist, and hence determines just how "fat" each Powerset is.

Problem z.1. Show that, for any sets A, B: (i) the set of all relations with domain A and range B exists; and (ii) the set of all functions from A to B exists.

Problem z.2. Let A be a set, and let $\sim$ be an equivalence relation on A. Prove that the set of equivalence classes under $\sim$ on A, i.e., $A/_{\sim}$, exists.

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Bibliography