z.1 Selecting our Natural Numbers

sth:z:nat:sec In **??**, we explicitly defined the expression "natural numbers". How should you understand this stipulation? It is not a metaphysical claim, but just a decision to *treat* certain sets as the natural numbers. We touched upon reasons for thinking this in **??**, **??** and **??**. But we can make these reasons even more pointed.

Our Axiom of Infinity follows von Neumann (1925). But here is another axiom, which we could have adopted instead:

> *Zermelo's 1908 Axiom of Infinity.* There is a set A such that $\emptyset \in A$ and $(\forall x \in A)\{x\} \in A$.

Had we used Zermelo's axiom, instead of our (von Neumann-inspired) Axiom of Infinity, we would equally well have been given a Dedekind infinite set, and so a Dedekind algebra. On Zermelo's approach, the distinguished element of our algebra would again have been $\emptyset$ (our surrogate for 0), but the injection would have been given by the map $x \mapsto \{x\}$, rather than $x \mapsto x \cup \{x\}$. The simplest upshot of this is that Zermelo treats 2 as $\{\{\emptyset\}\}$, whereas we (with von Neumann) treat 2 as $\{\emptyset, \{\emptyset\}\}$.

Why choose one axiom of Infinity rather than the other? The main practical reason is that von Neumann's approach "scales up" to handle transfinite numbers rather well. We will explore this from **??** onwards. However, from the simple perspective of *doing arithmetic*, both approaches would do equally well. So if someone tells you that the natural numbers *are* sets, the obvious question is: *Which sets are they?*

This precise question was made famous by Benacerraf (1965). But it is worth emphasising that it is just the most famous example of a phenomenon that we have encountered many times already. The basic point is this. Set theory gives us a way to *simulate* a bunch of "intuitive" kinds of entities: the reals, rationals, integers, and naturals, yes; but also ordered pairs, functions, and relations. However, set theory never provides us with a *unique* choice of simulation. There are *always* alternatives which—straightforwardly—would have served us just as well.

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Bibliography

Benacerraf, Paul. 1965. What numbers could not be. *The Philosophical Review* 74(1): 47–73.

von Neumann, John. 1925. Eine Axiomatisierung der Mengenlehre. *Journal für die reine und angewandte Mathematik* 154: 219–40.

Zermelo, Ernst. 1908. Untersuchungen über die Grundlagen der Mengenlehre I. *Mathematische Annalen* 65: 261–81.

