

z.1 Infinity

sth:z:infinity-again:
sec

We already have enough axioms to ensure that there are infinitely many sets (if there are any). For suppose some set exists, and so \emptyset exists (by ??). Now for any set x , the set $x \cup \{x\}$ exists by ??. So, applying this a few times, we will get sets as follows:

0. \emptyset
1. $\{\emptyset\}$
2. $\{\emptyset, \{\emptyset\}\}$
3. $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$
4. $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$

and we can check that each of these sets is distinct.

We have started the numbering from 0, for a few reasons. But one of them is this. It is not that hard to check that the set we have labelled “ n ” has exactly n members, and (intuitively) is formed at the n th stage.

But. This gives us *infinitely many* sets, but it does not guarantee that there is an *infinite set*, i.e., a set with infinitely many members. And this really matters: unless we can find a (Dedekind) infinite set, we cannot construct a Dedekind algebra. But we want a Dedekind algebra, so that we can treat it as the set of natural numbers. (Compare ??.)

Importantly, the axioms we have laid down so far do *not* guarantee the existence of any infinite set. So we have to lay down a new axiom:

Axiom (Infinity). There is a set, I , such that $\emptyset \in I$ and $(\forall x \in I)x \cup \{x\} \in I$. $\exists I((\exists o \in I)\forall x(x \notin o) \wedge (\forall x \in I)(\exists s \in I)\forall z(z \in s \leftrightarrow (z \in x \vee z = x)))$

It is easy to see that the set I given to us by the Axiom of Infinity is Dedekind infinite. Its distinguished element is \emptyset , and the injection on I is given by $s(x) = x \cup \{x\}$. Now, ?? showed how to extract a Dedekind Algebra from a Dedekind infinite set; and we will treat this as our set of natural numbers. More precisely:

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defnomega

Definition z.1. Let I be any set given to us by the Axiom of Infinity. Let s be the function $s(x) = x \cup \{x\}$. Let $\omega = \text{clo}_s(\emptyset)$. We call the members of ω the *natural numbers*, and say that n is the result of n -many applications of s to \emptyset .

You can now look back and check that the set labelled “ n ”, a few paragraphs earlier, will be treated *as* the number n .

We will discuss this significance of this stipulation in ??. For now, it enables us to prove an intuitive result:

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naturalnumbersarentinfinite

Proposition z.2. *No natural number is Dedekind infinite.*

Proof. The proof is by induction, i.e., ???. Clearly $0 = \emptyset$ is not Dedekind infinite. For the induction step, we will establish the contrapositive: if (absurdly) $s(n)$ is Dedekind infinite, then n is Dedekind infinite.

So suppose that $s(n)$ is Dedekind infinite, i.e., there is some **injection** f with $\text{ran}(f) \subsetneq \text{dom}(f) = s(n) = n \cup \{n\}$. There are two cases to consider.

Case 1: $n \notin \text{ran}(f)$. Let $g = f \upharpoonright_n$. Now $\text{ran}(f) \subseteq n$, and $f(n) \in n$, so that $\text{ran}(g) = \text{ran}(f) \setminus \{f(n)\} \subsetneq n = \text{dom}(g)$. Hence n is Dedekind infinite.

Case 2: $n \in \text{ran}(f)$. Fix $m \in \text{dom}(f) \setminus \text{ran}(f)$, and define a function f^* :

$$f^*(x) = \begin{cases} f(x) & \text{if } x \neq f^{-1}(n) \\ m & \text{if } x = f^{-1}(n) \end{cases}$$

So f^* and f agree everywhere, except that $f^*(f^{-1}(n)) = m \neq n = f(f^{-1}(n))$. Since f is an **injection**, $n \notin \text{ran}(f^*)$; and $\text{ran}(f^*) \subsetneq \text{dom}(f^*) = s(n)$. Now n is Dedekind infinite, as in Case 1. \square

The question remains, though, of how we might *justify* the Axiom of Infinity. The short answer is that we will need to another principle to the story we have been telling. That principle is as follows:

Stages-hit-infinity. There is an infinite stage. That is, there is a stage which (a) is not the first stage, and which (b) has some stages before it, but which (c) has no immediate predecessor.

The Axiom of Infinity follows straightforwardly from this principle. We know that natural number n is formed at stage n . So the set ω is formed at the first infinite stage. And ω itself witnesses the Axiom of Infinity.

This, however, simply pushes us back to the question of how we might justify *Stages-hit-infinity*. As with *Stages-keep-going*, it was not an explicit part of the story we told about the cumulative-iterative hierarchy. But more than that: nothing in the very idea of an iterative hierarchy, in which sets are formed stage by stage, forces us to think that the process involves an *infinite* stage. It seems perfectly coherent to think that the stages are ordered like the natural numbers.

This, however, gives rise to an obvious problem. In ??, we considered Dedekind's "proof" that there is a Dedekind infinite set (of thoughts). This may not have struck you as very satisfying. But if *Stages-hit-infinity* is not "forced upon us" by the iterative conception of set (or by "the laws of thought"), then we are still left without an intrinsic justification for the claim that there is a Dedekind infinite set.

There is much more to say here, of course. But hopefully you are now at a point to start thinking about what it might *take* to justify an axiom (or principle). In what follows we will simply take *Stages-hit-infinity* for granted.

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Bibliography