

story.1 Predicative and Impredicative

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The Russell set, R , was defined via $\{x : x \notin x\}$. Spelled out more fully, R would be the set which contains all and only those sets which are not non-self-membered. So in defining R , we quantify over the domain which would contain R (if it existed).

This is an *impredicative* definition. More generally, we might say that a definition is impredicative iff it quantifies over a domain which contains the object that is being defined.

In the wake of the paradoxes, Whitehead, Russell, Poincaré and Weyl rejected such impredicative definitions as “viciously circular”:

An analysis of the paradoxes to be avoided shows that they all result from a kind of vicious circle. The vicious circles in question arise from supposing that a collection of objects may contain members which can only be defined by means of the collection as a whole[. . .

¶]

The principle which enables us to avoid illegitimate totalities may be stated as follows: ‘Whatever involves *all* of a collection must not be one of the collection’; or, conversely: ‘If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total.’ We shall call this the ‘vicious-circle principle,’ because it enables us to avoid the vicious circles involved in the assumption of illegitimate totalities. (Whitehead and Russell, 1910, p. 37)

If we follow them in rejecting the *vicious-circle principle*, then we might attempt to replace the disastrous Naïve Comprehension Scheme (of ??) with something like this:

Predicative Comprehension. For every formula φ quantifying only over sets: the set' $\{x : \varphi(x)\}$ exists.

So long as sets' are not sets, no contradiction will ensue.

Unfortunately, Predicative Comprehension is not very *comprehensive*. After all, it introduces us to new entities, sets'. So we will have to consider formulas which quantify over sets'. If they always yield a set', then Russell's paradox will arise again, just by considering the set' of all non-self-membered sets'. So, pursuing the same thought, we must say that a formula quantifying over sets' yields a corresponding set''. And then we will need sets''', sets''', etc. To prevent a rash of primes, it will be easier to think of these as sets₀, sets₁, sets₂, sets₃, sets₄, . . . And this would give us a way into the (simple) theory of types.

There are a few obvious objections against such a theory (though it is not obvious that they are *overwhelming* objections). In brief: the resulting theory is cumbersome to use; it is profligate in postulating different kinds of objects;

and it is not clear, in the end, that impredicative definitions are even *all that bad*.

To bring out the last point, consider this remark from Ramsey:

we may refer to a man as the tallest in a group, thus identifying him by means of a totality of which he is himself a member without there being any vicious circle. (Ramsey, 1925)

Ramsey’s point is that “the tallest man in the group” *is* an impredicative definition; but it is obviously perfectly kosher.

One might respond that, in this case, we could pick out the tallest person by *predicative* means. For example, maybe we could just point at the man in question. The objection against impredicative definitions, then, would clearly need to be limited to entities which can *only* be picked out impredicatively. But even then, we would need to hear more, about why such “essential impredicativity” would be so bad.¹

Admittedly, impredicative definitions are extremely bad news, if we want our definitions to provide us with something like a recipe for *creating* an object. For, given an impredicative definition, one would genuinely be caught in a vicious circle: to create the impredicatively specified object, one would *first* need to create all the objects (including the impredicatively specified object), since the impredicatively specified object is specified in terms of all the objects; so one would need to create the impredicatively specified object before one had created it itself. But again, this is only a serious objection against “essentially impredicatively” specified sets, if we think of sets as things that we *create*. And we (probably) don’t.

As such—for better or worse—the approach which became common does not involve taking a hard line concerning (im)predicativity. Rather, it involves what is now regarded as the cumulative-iterative approach. In the end, this will allow us to stratify our sets into “stages”—a *bit* like the predicative approach stratifies entities into sets₀, sets₁, sets₂, ...—but we will not postulate any difference in kind between them.

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Bibliography

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¹For more, see Linnebo (2010).

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