Appendix: Frege’s Basic Law V

In ??, we explained that Russell’s formulated his paradox as a problem for the system Frege outlined in his Grundgesetze. Frege’s system did not include a direct formulation of Naïve Comprehension. So, in this appendix, we will very briefly explain what Frege’s system did include, and how it relates to Naïve Comprehension and how it relates to Russell’s Paradox.

Frege’s system is second-order, and was designed to formulate the notion of an extension of a concept. Using notation inspired by Frege, we will write \( \epsilon x F \) for the extension of the concept \( F \). This is a device which takes a predicate, “\( F \)”, and turns it into a (first-order) term, “\( \epsilon x F \)”.

Using this device, Frege offered the following definition of membership:

\[
a \in b \iff \exists G (\epsilon x F = \epsilon x G(x) \land Ga)
\]

roughly: \( a \in b \) iff \( a \) falls under a concept whose extension is \( b \). (Note that the quantifier “\( \exists G \)” is second-order.) Frege also maintained the following principle, known as Basic Law V:

\[
\epsilon x F(x) = \epsilon x G(x) \iff \forall x (Fx \leftrightarrow Gx)
\]

roughly: concepts have identical extensions iff they are coextensive. (Again, both “\( F \)” and “\( G \)” are in predicate position.) Now a simple principle connects membership with property-satisfaction:

**Lemma story.1 (in Grundgesetze).** \( \forall F \forall a (a \in \epsilon x F(x) \leftrightarrow Fa) \)

*Proof.* Fix \( F \) and \( a \). Now \( a \in \epsilon x F(x) \) iff \( \exists G (\epsilon x F(x) = \epsilon x G(x) \land Ga) \) (by the definition of membership) iff \( \exists G (\forall x (Fx \leftrightarrow Gx) \land Ga) \) (by Basic Law V) iff \( Fa \) (by elementary second-order logic). \( \square \)

And this yields Naïve Comprehension almost immediately:

**Lemma story.2 (in Grundgesetze).** \( \forall F \exists s \forall a (a \in s \leftrightarrow Fa) \)

*Proof.* Fix \( F \); now Lemma story.1 yields \( \forall a (a \in \epsilon x F(x) \leftrightarrow Fa) \); so \( \exists s \forall a (a \in s \leftrightarrow Fa) \) by existential generalisation. The result follows since \( F \) was arbitrary. \( \square \)

Russell’s Paradox follows by taking \( F \) as given by \( \forall x (Fx \leftrightarrow x \notin x) \).

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Bibliography