story.1 Appendix: Frege's Basic Law V

sth:story:blv: sec In ??, we explained that Russell's formulated his paradox as a problem for the system Frege outlined in his *Grundgesetze*. Frege's system did not include a direct formulation of Naïve Comprehension. So, in this appendix, we will very briefly explain what Frege's system *did* include, and how it relates to Naïve Comprehension and how it relates to Russell's Paradox.

Frege's system is *second-order*, and was designed to formulate the notion of an *extension of a concept*.[1] Using notation inspired by Frege, we will write $\epsilon x\, F(x)$ for *the extension of the concept F*. This is a device which takes a *predicate*, "F", and turns it into a (first-order) *term*, "$\epsilon x\, F(x)$". Using this device, Frege offered the following *definition* of membership:

$$a \in b =_{\text{df}} \exists G(b = \epsilon x\, G(x) \land Ga)$$

roughly: $a \in b$ iff a falls under a concept whose extension is b. (Note that the quantifier "$\exists G$" is second-order.) Frege also maintained the following principle, known as *Basic Law V*:

$$\epsilon x\, F(x) = \epsilon x\, G(x) \leftrightarrow \forall x(Fx \leftrightarrow Gx)$$

roughly: concepts have identical extensions iff they are coextensive. (Again, both "F" and "G" are in predicate position.) Now a simple principle connects membership with property-satisfaction:

sth:story:blv: lem:Fregeextensions **Lemma story.1 (in *Grundgesetze*).** $\forall F \forall a(a \in \epsilon x\, F(x) \leftrightarrow Fa)$

Proof. Fix F and a. Now $a \in \epsilon x\, F(x)$ iff $\exists G(\epsilon x\, F(x) = \epsilon x\, G(x) \land Ga)$ (by the definition of membership) iff $\exists G(\forall x(Fx \leftrightarrow Gx) \land Ga)$ (by Basic Law V) iff Fa (by elementary second-order logic). □

And this yields Naïve Comprehension almost immediately:

Lemma story.2 (in *Grundgesetze*.). $\forall F \exists s \forall a(a \in s \leftrightarrow Fa)$

Proof. Fix F; now Lemma story.1 yields $\forall a(a \in \epsilon x\, F(x) \leftrightarrow Fa)$; so $\exists s \forall a(a \in s \leftrightarrow Fa)$ by existential generalisation. The result follows since F was arbitrary. □

Russell's Paradox follows by taking F as given by $\forall x(Fx \leftrightarrow x \notin x)$.

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Bibliography

Heck, Richard Kimberly. 2012. *Reading Frege's Grundgesetze*. Oxford: Oxford University Press.

[1] Strictly speaking, Frege attempts to formalize a more general notion: the "value-range" of a function. Extensions of concepts are a special case of the more general notion. See Heck (2012, pp. 8–9) for the details.