story.1 The Cumulative-Iterative Approach

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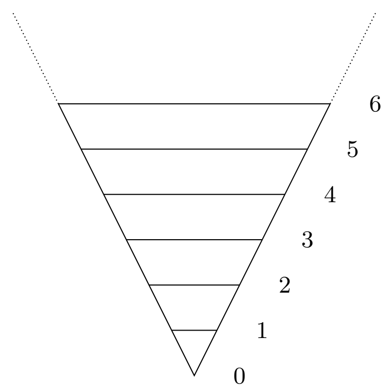
Here is a slightly fuller statement of how we will stratify sets into stages:

> Sets are formed in *stages*. For each stage S, there are certain stages which are *before* S. At stage S, each collection consisting of sets formed at stages before S is formed into a set. There are no sets other than the sets which are formed at stages. (Shoenfield, 1977, p. 323)

This is a sketch of the *cumulative-iterative conception of set*. It will underpin the formal set theory that we present in **??**.

Let's explore this in a little more detail. As Shoenfield describes the process, at every stage, we form new sets from the sets which were available to us from earlier stages. So, on Shoenfield's picture, at the initial stage, stage 0, there are no *earlier* stages, and so *a fortiori* there are no sets available to us from earlier stages.[1] So we form only one set: the set with no elements $\emptyset$. At stage 1, exactly one set is available to us from earlier stages, so only one new set is $\{\emptyset\}$. At stage 2, two sets are available to us from earlier stages, and we form two new sets $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$. At stage 3, four sets are available to us from earlier stages, so we form twelve new sets. . . . As such, the cumulative-iterative picture of the sets will look a bit like this (with numbers indicating stages):



So: why should we embrace this story?

One reason is that it is a nice, tractable story. Given the demise of the most obvious story, i.e., Naïve Comprehension, we are in want of something nice.

But the story is not *just* nice. We have a good reason to believe that any set theory based on this story will be *consistent*. Here is why.

Given the cumulative-iterative conception of set, we form sets at stages; and their elements must be objects which were available *already*. So, for any stage S, we can form the set

$$R_S = \{x : x \notin x \text{ and } x \text{ was available before } S\}$$

[1] Why should we assume that there *is* a first stage? See the footnote to *Stages-are-ordered* in **??**.

The reasoning involved in proving Russell's Paradox will now establish that R_S itself is not available before stage S. And that's not a contradiction. Moreover, if we embrace the cumulative-iterative conception of set, then we shouldn't even have *expected* to be able to form the Russell set itself. For that would be the set of all non-self-membered sets that "will ever be available". In short: the fact that we (provably) can't form the Russell set isn't *surprising*, given the cumulative-iterative story; it's what we would *predict*.

Photo Credits

Bibliography

Shoenfield, Joseph R. 1977. Axioms of set theory. In *Handbook of Mathematical Logic*, ed. Jon Barwise, 321–44. London: North-Holland.