

spine.1 Basic Properties of Stages

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To bring out the foundational importance of the definition of the V_α s, we will present a few basic results about them. We start with a definition:¹

Definition spine.1. The set A is *potent* iff $\forall x((\exists y \in A)x \subseteq y \rightarrow x \in A)$.

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Lemma spine.2. For each ordinal α :

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1. Each V_α is transitive.

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2. Each V_α is potent.

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3. If $\gamma \in \alpha$, then $V_\gamma \in V_\alpha$ (and hence also $V_\gamma \subseteq V_\alpha$ by (1))

Proof. We prove this by a (simultaneous) transfinite induction. For induction, suppose that (1)–(3) holds for each ordinal $\beta < \alpha$.

The case of $\alpha = \emptyset$ is trivial.

Suppose $\alpha = \beta^+$. To show (3), if $\gamma \in \alpha$ then $V_\gamma \subseteq V_\beta$ by hypothesis, so $V_\gamma \in \wp(V_\beta) = V_\alpha$. To show (2), suppose $A \subseteq B \in V_\alpha$ i.e., $A \subseteq B \subseteq V_\beta$; then $A \subseteq V_\beta$ so $A \in V_\alpha$. To show (1), note that if $x \in A \in V_\alpha$ we have $A \subseteq V_\beta$, so $x \in V_\beta$, so $x \subseteq V_\beta$ as V_β is transitive by hypothesis, and so $x \in V_\alpha$.

Suppose α is a limit ordinal. To show (3), if $\gamma \in \alpha$ then $\gamma \in \gamma^+ \in \alpha$, so that $V_\gamma \in V_{\gamma^+}$ by assumption, hence $V_\gamma \in \bigcup_{\beta \in \alpha} V_\beta = V_\alpha$. To show (1) and (2), just observe that a union of transitive (respectively, potent) sets is transitive (respectively, potent). \square

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Lemma spine.3. For each ordinal α , $V_\alpha \notin V_\alpha$.

Proof. By transfinite induction. Evidently $V_\emptyset \notin V_\emptyset$.

If $V_{\alpha^+} \in V_{\alpha^+} = \wp(V_\alpha)$, then $V_{\alpha^+} \subseteq V_\alpha$; and since $V_\alpha \in V_{\alpha^+}$ by Lemma spine.2, we have $V_\alpha \in V_\alpha$. Conversely: if $V_\alpha \notin V_\alpha$ then $V_{\alpha^+} \notin V_{\alpha^+}$.

If α is a limit and $V_\alpha \in V_\alpha = \bigcup_{\beta \in \alpha} V_\beta$, then $V_\alpha \in V_\beta$ for some $\beta \in \alpha$; but then also $V_\beta \in V_\alpha$ so that $V_\beta \in V_\beta$ by Lemma spine.2 (twice). Conversely, if $V_\beta \notin V_\beta$ for all $\beta \in \alpha$, then $V_\alpha \notin V_\alpha$. \square

Corollary spine.4. For any ordinals α, β : $\alpha \in \beta$ iff $V_\alpha \in V_\beta$

Proof. Lemma spine.2 gives one direction. Conversely, suppose $V_\alpha \in V_\beta$. Then $\alpha \neq \beta$ by Lemma spine.3; and $\beta \notin \alpha$, for otherwise we would have $V_\beta \in V_\alpha$ and hence $V_\beta \in V_\beta$ by Lemma spine.2 (twice), contradicting Lemma spine.3. So $\alpha \in \beta$ by Trichotomy. \square

All of this allows us to think of each V_α as the α th stage of the hierarchy. Here is why.

Certainly our V_α s can be thought of as being formed in an *iterative* process, for our use of ordinals tracks the notion of iteration. Moreover, if one stage

¹There's no standard terminology for "potent"; this is the name used by Button (2021).

is formed before the other, i.e., $V_\beta \in V_\alpha$, i.e., $\beta \in \alpha$, then our process of formation is *cumulative*, since $V_\beta \subseteq V_\alpha$. Finally, we are indeed forming *all* possible collections of sets that were available at any earlier stage, since any successor stage $V_{\alpha+}$ is the power-set of its predecessor V_α .

In short: with \mathbf{ZF}^- , we are *almost* done, in articulating our vision of the cumulative-iterative hierarchy of sets. (Though, of course, we still need to justify Replacement.)

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Bibliography

Button, Tim. 2021. Level theory, part 1: Axiomatizing the bare idea of a cumulative hierarchy of sets. *The Bulletin of Symbolic Logic* 27(4): 436–460.