spine.1 Basic Properties of Stages

To bring out the foundational importance of the definition of the $V_\alpha$s, we will start with a few results about them.

Lemma spine.1. For each ordinal $\alpha$:

1. Each $V_\alpha$ is a transitive set.
2. Each $V_\alpha$ is a sublative set, \(^1\) i.e., $\forall A(\exists B(A \subseteq B \in V_\alpha) \implies A \in V_\alpha)$.
3. If $\gamma \in \alpha$, then $V_\gamma \subseteq V_\alpha$ (and hence also $V_\gamma \subseteq V_\alpha$ by (1))

Proof. We prove this by a (simultaneous) transfinite induction. For induction, suppose that (1)–(3) holds for each ordinal $\beta < \alpha$.

The case of $\alpha = \emptyset$ is trivial.

Suppose $\alpha = \beta^+$. To show (3), if $\gamma \in \alpha$ then $V_\gamma \subseteq V_\beta$ by hypothesis, so $V_\gamma \subseteq \varphi(V_\beta) = V_\alpha$. To show (2), suppose $A \subseteq B \in V_\alpha$ i.e., $A \subseteq B \subseteq V_\beta$; then $A \subseteq V_\beta$ so $A \in V_\alpha$. To show (1), note that if $x \in A \in V_\alpha$ we have $A \subseteq V_\beta$, so $x \in V_\beta$, so $x \subseteq V_\beta$ as $V_\beta$ is transitive by hypothesis, and so $x \in V_\alpha$.

Suppose $\alpha$ is a limit ordinal. To show (3), if $\gamma \in \alpha$ then $\gamma \in \gamma^+ \in \alpha$, so that $V_\gamma \in V_{\gamma^+}$ by assumption, hence $V_\gamma \in \bigcup_{\beta \in \alpha} V_\beta = V_\alpha$. To show (1) and (2), just observe that a union of transitive (respectively, sublative) sets is transitive (respectively, sublative).

Lemma spine.2. For each ordinal $\alpha$, $V_\alpha \notin V_\alpha$.

Proof. By transfinite induction. Evidently $V_\emptyset \notin V_\emptyset$.

If $V_{\alpha^+} \in V_{\alpha^+} = \varphi(V_\alpha)$, then $V_{\alpha^+} \subseteq V_\alpha$; and since $V_\alpha \in V_{\alpha^+}$ by Lemma spine.1, we have $V_\alpha \in V_\alpha$. Conversely: if $V_\alpha \notin V_\alpha$ then $V_{\alpha^+} \notin V_{\alpha^+}$.

If $\alpha$ is a limit and $V_\alpha \in V_\alpha = \bigcup_{\beta \in \alpha} V_\beta$, then $V_\alpha \in V_\beta$ for some $\beta \in \alpha$; but then also $V_\beta \in V_\alpha$, so that $V_\beta \in V_\beta$ by Lemma spine.1 (twice). Conversely, if $V_\beta \notin V_\beta$ for all $\beta \in \alpha$, then $V_\alpha \notin V_\alpha$.

Corollary spine.3. For any ordinals $\alpha, \beta$: $\alpha \in \beta$ iff $V_\alpha \in V_\beta$

Proof. Lemma spine.1 gives one direction. Conversely, suppose $V_\alpha \in V_\beta$. Then $\alpha \neq \beta$ by Lemma spine.2; and $\beta \notin \alpha$, for otherwise we would have $V_\beta \in V_\alpha$ and hence $V_\beta \in V_\beta$ by Lemma spine.1 (twice), contradicting Lemma spine.2. So $\alpha \in \beta$ by Trichotomy.

All of this allows us to think of each $V_\alpha$ as the $\alpha$th stage of the hierarchy. Here is why.

Certainly our $V_\alpha$s can be thought of as being formed in an iterative process, for our use of ordinals tracks the notion of iteration. Moreover, if one stage is formed before the other, i.e., $V_\beta \in V_\alpha$, i.e., $\beta \in \alpha$, then our process of formation is cumulative, since $V_\beta \subseteq V_\alpha$. Finally, we are indeed forming all

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\(^1\)There’s no standard terminology for “sublativity”. But this seems good.
possible collections of sets that were available at any earlier stage, since any successor stage $V_{\alpha +}$ is the power-set of its predecessor $V_\alpha$.

In short: with $\text{ZF}^-$, we are almost done, in articulating our vision of the cumulative-iterative hierarchy of sets. (Though, of course, we still need to justify Replacement.)

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Bibliography