

## spine.1 The Transfinite Recursion Theorem(s)

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The first thing we must do, though, is confirm that ?? is a successful definition. More generally, we need to prove that any attempt to offer a transfinite by (transfinite) recursion will succeed. That is the aim of this section.

*Warning: this is very tricky material.* The overarching moral, though, is quite simple: Transfinite Induction plus Replacement guarantee the legitimacy of (several versions of) transfinite recursion.

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**Theorem spine.1** (Bounded Recursion). *For any term  $\tau(x)$  and any ordinal  $\alpha$ ,<sup>1</sup> there is a unique function  $f$  with domain  $\alpha$  such that  $(\forall \beta \in \alpha) f(\beta) = \tau(f \upharpoonright_\beta)$*

*Proof.* We will show that, for any  $\delta \leq \alpha$ , there is a unique  $g_\delta$  with domain  $\delta$  such that  $(\forall \beta \in \delta) g(\beta) = \tau(g \upharpoonright_\beta)$ .

We first establish uniqueness. Given  $g_{\delta_1}$  and  $g_{\delta_2}$ , a transfinite induction on their arguments shows that  $g(\beta) = h(\beta)$  for any  $\beta \in \text{dom}(g) \cap \text{dom}(h) = \delta_1 \cap \delta_2 = \min(\delta_1, \delta_2)$ . So our functions are unique (if they exist), and agree on all values.

To establish existence, we now use a simple transfinite induction (??) on ordinals  $\delta \leq \alpha$ .

Let  $g_\emptyset = \emptyset$ ; this trivially behaves correctly.

Given  $g_\delta$ , let  $g_{\delta^+} = g_\delta \cup \{\langle \delta, \tau(g_\delta) \rangle\}$ . This behaves correctly as  $g_{\delta^+} \upharpoonright_\delta = g_\delta$ .

Given  $g_\gamma$  for all  $\gamma \leq \delta$  with  $\delta$  a limit ordinal, let  $g_\delta = \bigcup_{\gamma \in \delta} g_\gamma$ . This is a function, since our various  $g_\beta$ 's agree on all values. And if  $\beta \in \delta$  then  $g_\delta(\beta) = g_{\beta^+}(\beta) = \tau(g_{\beta^+} \upharpoonright_\beta) = \tau(g_\delta \upharpoonright_\beta)$ .

This completes the proof by transfinite induction. Now just let  $f = g_\alpha$ .  $\square$

If we allow ourselves to define a *term* rather than a function, then we can remove the bound  $\alpha$  from the previous result. (In the statement and proof of this result, when  $\sigma$  is a term, we let  $\sigma \upharpoonright_\alpha = \{\langle \gamma, \sigma(\gamma) \rangle : \gamma \in \alpha\}$ .)

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**Theorem spine.2** (General Recursion). *For any term  $\tau(x)$  we can explicitly define a term  $\sigma(x)$ ,<sup>2</sup> such that  $\sigma(\alpha) = \tau(\sigma \upharpoonright_\alpha)$  for any ordinal  $\alpha$ .*

*Proof.* For each  $\alpha$ , by **Theorem spine.1** are unique  $\alpha^+$ -approximations,  $f_{\alpha^+}$ , and:

$$f_{\alpha^+}(\alpha) = \tau(f_{\alpha^+} \upharpoonright_\alpha) = \tau(\{\langle \gamma, f_{\alpha^+}(\gamma) \rangle : \gamma \in \alpha\}).$$

So define  $\sigma(\alpha)$  as  $f_{\alpha^+}(\alpha)$ . Repeating the induction of **Theorem spine.1**, but without the upper bound, this is well-defined.  $\square$

Note that these results are *schemas*. Crucially, we cannot expect  $\sigma$  to define a function, i.e., a certain kind of *set*, since then  $\text{dom}(\sigma)$  would be the set of all ordinals, contradicting the Burali-Forti Paradox (??).

<sup>1</sup>The term may have parameters.

<sup>2</sup>Both terms may have parameters.

It still remains to show, though, that [Theorem spine.2](#) vindicates our definition of the  $V_\alpha$ s. This may not be immediately obvious; but it will become apparent with a last, simple, version of transfinite recursion.

**Theorem spine.3** (Simple Recursion). *For any terms  $\tau_1(x)$  and  $\tau_2(x)$  and any set  $A$ , we can explicitly define a term  $\sigma(x)$  such that:*<sup>3</sup> [sth:spine:recursion:simplerecursionschema](#)

$$\begin{aligned} \sigma(\emptyset) &= A \\ \sigma(\alpha^+) &= \tau_1(\sigma(\alpha)) && \text{for any ordinal } \alpha \\ \sigma(\alpha) &= \tau_2(\text{ran}(\sigma \upharpoonright_\alpha)) && \text{when } \alpha \text{ is a limit ordinal} \end{aligned}$$

*Proof.* We start by defining a term,  $\xi(x)$ , as follows:

$$\xi(x) = \begin{cases} A & \text{if } x \text{ is not a function whose domain is an ordinal;} \\ & \text{otherwise:} \\ \tau_1(x(\alpha)) & \text{if } \text{dom}(x) = \alpha^+ \\ \tau_2(\text{ran}(x)) & \text{if } \text{dom}(x) \text{ is a limit ordinal} \end{cases}$$

By [Theorem spine.2](#), there is a term  $\sigma(x)$  such that  $\sigma(\alpha) = \xi(\sigma \upharpoonright_\alpha)$  for every ordinal  $\alpha$ ; moreover,  $\sigma \upharpoonright_\alpha$  is a function with domain  $\alpha$ . We show that  $\sigma$  has the required properties, by simple transfinite induction (??).

First,  $\sigma(\emptyset) = \xi(\emptyset) = A$ .

Next,  $\sigma(\alpha^+) = \xi(\sigma \upharpoonright_{\alpha^+}) = \tau_1(\sigma \upharpoonright_{\alpha^+}(\alpha)) = \tau_1(\sigma(\alpha))$ .

Finally, if  $\alpha$  is a limit ordinal,  $\sigma(\alpha) = \xi(\sigma \upharpoonright_\alpha) = \tau_2(\text{ran}(\sigma \upharpoonright_\alpha))$ . □

Now, to vindicate ??, just take  $A = \emptyset$  and  $\tau_1(x) = \wp(x)$  and  $\tau_2(x) = \bigcup x$ . So we have vindicated the definition of the  $V_\alpha$ s!

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## Bibliography

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<sup>3</sup>The terms may have parameters.