spine.1  The Transfinite Recursion Theorem(s)

The first thing we must do, though, is confirm that ?? is a successful definition. More generally, we need to prove that any attempt to offer a transfinite by (transfinite) recursion will succeed. That is the aim of this section.

**Warning**: this is very tricky material. The overarching moral, though, is quite simple: Transfinite Induction plus Replacement guarantee the legitimacy of (several versions of) transfinite recursion.

**Theorem spine.1 (Bounded Recursion).** For any term \( \tau(x) \) and any ordinal \( \alpha \), \(^1\) there is a unique function \( f \) with domain \( \alpha \) such that \( (\forall \beta \in \alpha) f(\beta) = \tau(f|_\beta) \)

**Proof.** We will show that, for any \( \delta \leq \alpha \), there is a unique \( g_\delta \) with domain \( \delta \) such that \( (\forall \beta \in \delta) g_\delta(\beta) = \tau(g_\delta|_\beta) \).

We first establish uniqueness. Given \( g_\delta_1 \) and \( g_\delta_2 \), a transfinite induction on their arguments shows that \( g_\delta(\beta) = h(\beta) \) for any \( \beta \in \text{dom}(g) \cap \text{dom}(h) = \delta_1 \cap \delta_2 = \min(\delta_1, \delta_2) \). So our functions are unique (if they exist), and agree on all values.

To establish existence, we now use a simple transfinite induction (??) on ordinals \( \delta \leq \alpha \).

Let \( g_\emptyset = \emptyset \); this trivially behaves correctly.

Given \( g_\delta \), let \( g_{\delta^+} = g_\delta \cup \{ (\delta, \tau(g_\delta)) \} \). This behaves correctly as \( g_{\delta^+}|_\delta = g_\delta \).

Given \( g_\gamma \) for all \( \gamma \leq \delta \) with \( \delta \) a limit ordinal, let \( g_\delta = \bigcup_{\gamma \in \delta} g_\gamma \). This is a function, since our various \( g_\gamma \)'s agree on all values. And if \( \beta \in \delta \) then \( g_\delta(\beta) = g_{\delta^+}(\beta) = \tau(g_{\delta^+}|_\beta) = \tau(g_\delta|_\beta) \).

This completes the proof by transfinite induction. Now just let \( f = g_\alpha \). \( \square \)

If we allow ourselves to define a term rather than a function, then we can remove the bound \( \alpha \) from the previous result. (In the statement and proof of this result, when \( \sigma \) is a term, we let \( \sigma|_\alpha = \{ \langle \gamma, \sigma(\gamma) \rangle : \gamma \in \alpha \} \).)

**Theorem spine.2 (General Recursion).** For any term \( \tau(x) \) we can explicitly define a term \( \sigma(x) \), \(^2\) such that \( \sigma(\alpha) = \tau(\sigma|_\alpha) \) for any ordinal \( \alpha \).

**Proof.** For each \( \alpha \), by **Theorem spine.1** are unique \( \alpha^+ \)-approximations, \( f_{\alpha^+} \), and:

\[
f_{\alpha^+}(\alpha) = \tau(f_{\alpha^+}|_\alpha) = \tau(\{ (\gamma, f_{\alpha^+}(\gamma)) : \gamma \in \alpha \}).
\]

So define \( \sigma(\alpha) \) as \( f_{\alpha^+}(\alpha) \). Repeating the induction of **Theorem spine.1**, but without the upper bound, this is well-defined. \( \square \)

Note that these results are schemas. Crucially, we cannot expect \( \sigma \) to define a function, i.e., a certain kind of set, since then \( \text{dom}(\sigma) \) would be the set of all ordinals, contradicting the Burali-Forti Paradox (??).

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\(^1\)The term may have parameters.

\(^2\)Both terms may have parameters.
It still remains to show, though, that Theorem spine.2 vindicates our definition of the $V_\alpha$s. This may not be immediately obvious; but it will become apparent with a last, simple, version of transfinite recursion.

**Theorem spine.3 (Simple Recursion).** For any terms $\tau_1(x)$ and $\tau_2(x)$ and any set $A$, we can explicitly define a term $\sigma(x)$ such that:

\[
\begin{align*}
\sigma(\emptyset) &= A \\
\sigma(\alpha^+) &= \tau_1(\sigma(\alpha)) & \text{for any ordinal } \alpha \\
\sigma(\alpha) &= \tau_2(\text{ran}(\sigma|_\alpha)) & \text{when } \alpha \text{ is a limit ordinal}
\end{align*}
\]

**Proof.** We start by defining a term, $\xi(x)$, as follows:

\[
\xi(x) = \begin{cases} 
A & \text{if } x \text{ is not a function whose domain is an ordinal;} \\
\tau_1(x(\alpha)) & \text{if } \text{dom}(x) = \alpha^+ \\
\tau_2(\text{ran}(x)) & \text{if } \text{dom}(x) \text{ is a limit ordinal}
\end{cases}
\]

By Theorem spine.2, there is a term $\sigma(x)$ such that $\sigma(\alpha) = \xi(\sigma|_\alpha)$ for every ordinal $\alpha$; moreover, $\sigma|_\alpha$ is a function with domain $\alpha$. We show that $\sigma$ has the required properties, by simple transfinite induction (??).

First, $\sigma(\emptyset) = \xi(\emptyset) = A$.
Next, $\sigma(\alpha^+) = \xi(\sigma|_{\alpha^+}) = \tau_1(\sigma|_{\alpha^+}(\alpha)) = \tau_1(\sigma(\alpha))$.
Finally, if $\alpha$ is a limit ordinal, $\sigma(\alpha) = \xi(\sigma|_\alpha) = \tau_2(\text{ran}(\sigma|_\alpha))$. \qed

Now, to vindicate ??, just take $A = \emptyset$ and $\tau_1(x) = \wp(x)$ and $\tau_2(x) = \bigcup x$. So we have vindicated the definition of the $V_\alpha$s!

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**Bibliography**

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3 The terms may have parameters.