spine.1 Rank

Now that we have defined the stages as the $V_\alpha$'s, and we know that every set is a subset of some stage, we can define the rank of a set. Intuitively, the rank of $A$ is the first moment at which $A$ is formed. More precisely:

**Definition spine.1.** For each set $A$, $\text{rank}(A)$ is the least ordinal $\alpha$ such that $A \subseteq V_\alpha$.

**Proposition spine.2.** $\text{rank}(A)$ exists, for any $A$.

*Proof.* Left as an exercise. \qed

**Problem spine.1.** Prove Proposition spine.2.

The well-ordering of ranks allows us to prove some important results:

**Proposition spine.3.** For any ordinal $\alpha$, $V_\alpha = \{ x : \text{rank}(x) \in \alpha \}$.

*Proof.* If rank($x$) $\in \alpha$ then $x \subseteq V_{\text{rank}(x)} \subseteq V_\alpha$, so $x \in V_\alpha$ as $V_\alpha$ is potent (invoking ?? multiple times). Conversely, if $x \in V_\alpha$ then $x \subseteq V_\alpha$, so rank($x$) $\leq \alpha$; now a simple transfinite induction shows that $x \notin V_\alpha$. \qed

**Problem spine.2.** Complete the simple transfinite induction mentioned in Proposition spine.3.

**Proposition spine.4.** If $B \in A$, then $\text{rank}(B) \in \text{rank}(A)$.

*Proof.* $A \subseteq V_{\text{rank}(A)} = \{ x : \text{rank}(x) \in \text{rank}(A) \}$ by Proposition spine.3. \qed

Using this fact, we can establish a result which allows us to prove things about all sets by a form of induction:

**Theorem spine.5 (\in-Induction Scheme).** For any formula $\varphi$:

$$\forall A((\forall x \in A) \varphi(x) \rightarrow \varphi(A)) \rightarrow \forall A \varphi(A).$$

*Proof.* We will prove the contrapositive. So, suppose $\neg \forall A \varphi(A)$. By Transfinite Induction (??), there is some non-$\varphi$ of least possible rank; i.e. some $A$ such that $\neg \varphi(A)$ and $\forall x (\text{rank}(x) \in \text{rank}(A) \rightarrow \varphi(x))$. Now if $x \in A$ then $\text{rank}(x) \in \text{rank}(A)$, by Proposition spine.4, so that $\varphi(x)$; i.e. $(\forall x \in A) \varphi(x) \land \neg \varphi(A)$. \qed

Here is an informal way to gloss this powerful result. Say that $\varphi$ is *hereditary* iff whenever every element of a set is $\varphi$, the set itself is $\varphi$. Then $\in$-Induction tells you the following: if $\varphi$ is hereditary, every set is $\varphi$.

To wrap up the discussion of ranks (for now), we’ll prove a few claims which we have foreshadowed a few times.

**Proposition spine.6.** $\text{rank}(A) = \bigcup_{x \in A} \text{rank}(x)$. 

*Proof.* c9d2ed6 (2023-09-14) by OLP / CC–BY
Proof. Let \( \alpha = \text{lsub}_{x \in A} \text{rank}(x) \). By Proposition spine.4, \( \alpha \leq \text{rank}(A) \). But if \( x \in A \) then \( \text{rank}(x) \in \alpha \), so that \( x \in V_\alpha \) by Proposition spine.3, and hence \( A \subseteq V_\alpha \), i.e., \( \text{rank}(A) \leq \alpha \). Hence \( \text{rank}(A) = \alpha \). \qed

**Corollary spine.7.** For any ordinal \( \alpha \), \( \text{rank}(\alpha) = \alpha \).

Proof. Suppose for transfinite induction that \( \text{rank}(\beta) = \beta \) for all \( \beta \in \alpha \). Now
\[
\text{rank}(\alpha) = \text{lsub}_{\beta \in \alpha} \text{rank}(\beta) = \text{lsub}_{\beta \in \alpha} \beta = \alpha
\]
by Proposition spine.6. \qed

Finally, here is a quick proof of the result promised at the end of ??, that ZF\(^-\) proves the conditional Regularity \( \Rightarrow \) Foundation. (Note that the notion of “rank” and Proposition spine.4 are available for use in this proof since—as mentioned at the start of this section—they can be presented using ZF\(^-\) + Regularity.)

**Proposition spine.8 (working in ZF\(^-\) + Regularity).** Foundation holds.

Proof. Fix \( A \neq \emptyset \), and some \( B \in A \) of least possible rank. If \( c \in B \) then \( \text{rank}(c) \in \text{rank}(B) \) by Proposition spine.4, so that \( c \notin A \) by choice of \( B \). \qed

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Bibliography