

## spine.1 Rank

sth:spine:rank:sec Now that we have defined the stages as the  $V_\alpha$ 's, and we know that every set is a subset of some stage, we can define the *rank* of a set. Intuitively, the rank of  $A$  is the first moment at which  $A$  is formed. More precisely:

sth:spine:rank:dfnsetrank **Definition spine.1.** For each set  $A$ ,  $\text{rank}(A)$  is the least ordinal  $\alpha$  such that  $A \subseteq V_\alpha$ .

sth:spine:rank:ranksexist **Proposition spine.2.**  $\text{rank}(A)$  exists, for any  $A$ .

*Proof.* Left as an exercise. □

**Problem spine.1.** Prove **Proposition spine.2**.

The well-ordering of ranks allows us to prove some important results:

sth:spine:rank:valphalowerank **Proposition spine.3.** For any ordinal  $\alpha$ ,  $V_\alpha = \{x : \text{rank}(x) \in \alpha\}$ .

*Proof.* If  $\text{rank}(x) \in \alpha$  then  $x \subseteq V_{\text{rank}(x)} \in V_\alpha$ , so  $x \in V_\alpha$  as  $V_\alpha$  is potent (invoking ?? multiple times). Conversely, if  $x \in V_\alpha$  then  $x \subseteq V_\alpha$ , so  $\text{rank}(x) \leq \alpha$ ; now a simple transfinite induction shows that  $x \notin V_\alpha$ . □

**Problem spine.2.** Complete the simple transfinite induction mentioned in **Proposition spine.3**.

sth:spine:rank:rankmemberslower **Proposition spine.4.** If  $B \in A$ , then  $\text{rank}(B) \in \text{rank}(A)$ .

*Proof.*  $A \subseteq V_{\text{rank}(A)} = \{x : \text{rank}(x) \in \text{rank}(A)\}$  by **Proposition spine.3**. □

Using this fact, we can establish a result which allows us to prove things about *all sets* by a form of induction:

**Theorem spine.5 ( $\in$ -Induction Scheme).** For any formula  $\varphi$ :

$$\forall A((\forall x \in A)\varphi(x) \rightarrow \varphi(A)) \rightarrow \forall A\varphi(A).$$

*Proof.* We will prove the contrapositive. So, suppose  $\neg\forall A\varphi(A)$ . By Transfinite Induction (??), there is some non- $\varphi$  of least possible rank; i.e. some  $A$  such that  $\neg\varphi(A)$  and  $\forall x(\text{rank}(x) \in \text{rank}(A) \rightarrow \varphi(x))$ . Now if  $x \in A$  then  $\text{rank}(x) \in \text{rank}(A)$ , by **Proposition spine.4**, so that  $\varphi(x)$ ; i.e.  $(\forall x \in A)\varphi(x) \wedge \neg\varphi(A)$ . □

Here is an informal way to gloss this powerful result. Say that  $\varphi$  is *hereditary* iff whenever every *element* of a set is  $\varphi$ , the set itself is  $\varphi$ . Then  $\in$ -Induction tells you the following: if  $\varphi$  is hereditary, every set is  $\varphi$ .

To wrap up the discussion of ranks (for now), we'll prove a few claims which we have foreshadowed a few times.

sth:spine:rank:ranksupstrict **Proposition spine.6.**  $\text{rank}(A) = \text{lsub}_{x \in A} \text{rank}(x)$ .

*Proof.* Let  $\alpha = \text{lsub}_{x \in A} \text{rank}(x)$ . By [Proposition spine.4](#),  $\alpha \leq \text{rank}(A)$ . But if  $x \in A$  then  $\text{rank}(x) \in \alpha$ , so that  $x \in V_\alpha$  by [Proposition spine.3](#), and hence  $A \subseteq V_\alpha$ , i.e.,  $\text{rank}(A) \leq \alpha$ . Hence  $\text{rank}(A) = \alpha$ .  $\square$

**Corollary spine.7.** *For any ordinal  $\alpha$ ,  $\text{rank}(\alpha) = \alpha$ .*

[sth:spine:rank:ordsetrankalpha](#)

*Proof.* Suppose for transfinite induction that  $\text{rank}(\beta) = \beta$  for all  $\beta \in \alpha$ . Now  $\text{rank}(\alpha) = \text{lsub}_{\beta \in \alpha} \text{rank}(\beta) = \text{lsub}_{\beta \in \alpha} \beta = \alpha$  by [Proposition spine.6](#).  $\square$

Finally, here is a quick proof of the result promised at the end of ??, that  $\mathbf{ZF}^-$  proves the conditional *Regularity*  $\Rightarrow$  *Foundation*. (Note that the notion of “rank” and [Proposition spine.4](#) are available for use in this proof since—as mentioned at the start of this section—they can be presented using  $\mathbf{ZF}^- + \text{Regularity}$ .)

**Proposition spine.8 (working in  $\mathbf{ZF}^- + \text{Regularity}$ ).** *Foundation holds.*

[sth:spine:rank:zfminusregularityfoundation](#)

*Proof.* Fix  $A \neq \emptyset$ , and some  $B \in A$  of least possible rank. If  $c \in B$  then  $\text{rank}(c) \in \text{rank}(B)$  by [Proposition spine.4](#), so that  $c \notin A$  by choice of  $B$ .  $\square$

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## Bibliography