Defining the Stages as the $V_\alpha$s

In ??, we defined well-orderings and the (von Neumann) ordinals. In this chapter, we will use these to characterise the hierarchy of sets itself. To do this, recall that in ??, we defined the idea of successor and limit ordinals. We use these ideas in following definition:

**Definition spine.1.**

\[
\begin{align*}
V_0 &= \emptyset \\
V_\alpha^+ &= \wp(V_\alpha) \quad \text{for any ordinal } \alpha \\
V_\alpha &= \bigcup_{\gamma < \alpha} V_\gamma \quad \text{when } \alpha \text{ is a limit ordinal}
\end{align*}
\]

This will be a definition by *transfinite recursion* on the ordinals. In this regard, we should compare this with recursive definitions of functions on the natural numbers.\(^1\) As when dealing with natural numbers, one defines a base case and successor cases; but when dealing with ordinals, we also need to describe the behaviour of *limit* cases.

This definition of the $V_\alpha$s will be an important milestone. We have informally motivated our hierarchy of sets as forming sets by *stages*. The $V_\alpha$s are, in effect, just those stages. Importantly, though, this is an *internal* characterisation of the stages. Rather than suggesting a possible *model* of the theory, we will have defined the stages *within* our set theory.

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**Bibliography**

\(^1\)Cf. the definitions of addition, multiplication, and exponentiation in ??.