spine.1 Foundation

We have almost articulated the vision of the iterative-cumulative hierarchy in ZF. “Almost”, because there is a wrinkle. Nothing in ZF guarantees that every set is in some $V_\alpha$, i.e., that every set is formed at some stage.

Now, there is a fairly straightforward (mathematical) sense in which we don’t care whether there are sets outside the hierarchy. (If there are any there, we can simply ignore them.) But we have motivated our concept of set with the thought that every set is formed at some stage (see Stages-are-key in ?). So we will want to preclude the possibility of sets which fall outside of the hierarchy. Accordingly, we must add a new axiom, which ensures that every set occurs somewhere in the hierarchy.

Since the $V_\alpha$s are our stages, we might simply consider adding the following as an axiom:

Regularity. $\forall A \exists \alpha A \subseteq V_\alpha$

This is von Neumann’s approach (1925). However, for reasons that will be explained in the next section, we will instead adopt an alternative axiom:

Axiom (Foundation). $(\forall A \neq \emptyset)(\exists B \in A) A \cap B = \emptyset$.

With some effort, we can show (in ZF) that Foundation entails Regularity:

Definition spine.1. For each set $A$, let:

$$
\begin{align*}
\text{cl}_0(A) &= A, \\
\text{cl}_{n+1}(A) &= \bigcup \text{cl}_n(A), \\
\text{trcl}(A) &= \bigcup_{n<\omega} \text{cl}(A).
\end{align*}
$$

We call trcl$(A)$ the transitive closure of $A$. The name is apt:

Proposition spine.2. $A \subseteq \text{trcl}(A)$ and trcl$(A)$ is a transitive set.

Proof. Evidently $A = \text{cl}_0(A) \subseteq \text{trcl}(A)$. And if $x \in b \in \text{trcl}(A)$, then $b \in \text{cl}_n(A)$ for some $n$, so $x \in \text{cl}_{n+1}(A) \subseteq \text{trcl}(A)$.

Lemma spine.3. If $A$ is a transitive set, then there is some $\alpha$ such that $A \subseteq V_\alpha$.

Proof. Recalling the definition of “lsub$(X)$” from ??, define:

$$
\begin{align*}
D &= \{ x \in A : \forall \delta x \not\subseteq V_\delta \} \\
\alpha &= \text{lsub}\{ \delta : (\exists x \in A) (x \subseteq V_\delta \land (\forall \gamma \in \delta) x \not\subseteq V_\gamma) \}
\end{align*}
$$
Suppose $D = \emptyset$. So if $x \in A$, then there is some $\delta \in \alpha$ such that $x \subseteq V_\delta$, so $x \in V_\alpha$ by ???. Hence $A \subseteq V_\alpha$, as required.

So it suffices to show that $D = \emptyset$. For reductio, suppose otherwise. By Foundation, there is some $B \in D$ such that $D \cap B = \emptyset$. If $x \in B$ then $x \in A$, since $A$ is transitive, and since $x \notin D$, it follows that $\exists \delta \ x \subseteq V_\delta$. So now let

$$\beta = \{ \delta : (\exists x \in b)(x \subseteq V_\delta \land (\forall \gamma < \delta)x \notin V_\gamma) \}.$$ 

As before, $B \subseteq V_\beta$, contradicting the claim that $B \in D$. \qed

**Theorem spine.4.** Regularity holds.

**Proof.** Fix $A$; now $A \subseteq \text{trcl}(A)$ by Proposition spine.2, which is transitive. So there is some $\alpha$ such that $A \subseteq \text{trcl}(A) \subseteq V_\alpha$ by Lemma spine.3 \qed

These results show that $\text{ZF}^-$ proves the conditional $\text{Foundation} \Rightarrow \text{Regularity}$. In ??, we will show that $\text{ZF}^-$ proves $\text{Regularity} \Rightarrow \text{Foundation}$. As such, Foundation and Regularity are equivalent (modulo $\text{ZF}^-$). But this means that, given $\text{ZF}^-$, we can justify Foundation by noting that it is equivalent to Regularity. And we can justify Regularity immediately on the basis of Stages-are-key.

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**Bibliography**